APR 5 1949

Mathematical Reviews

Edited by

W. Feller **M. H. Stone** **H. Whitney**

R. P. Boas, Jr., *Executive Editor*

Vol. 10, No. 2 February, 1949 pp. 93–172

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MATHEMATICAL REVIEWS

Published monthly, except August, by

THE AMERICAN MATHEMATICAL SOCIETY, Prince and Lemon Streets, Lancaster, Pennsylvania

Sponsored by

THE AMERICAN MATHEMATICAL SOCIETY
THE MATHEMATICAL ASSOCIATION OF AMERICA
THE INSTITUTE OF MATHEMATICAL STATISTICS
THE EDINBURGH MATHEMATICAL SOCIETY
L'INTERMÉDIAIRE DES RECHERCHES MATHÉMATIQUES
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THE LONDON MATHEMATICAL SOCIETY
POLISH MATHEMATICAL SOCIETY
UNIÓN MATEMÁTICA ARGENTINA
INDIAN MATHEMATICAL SOCIETY

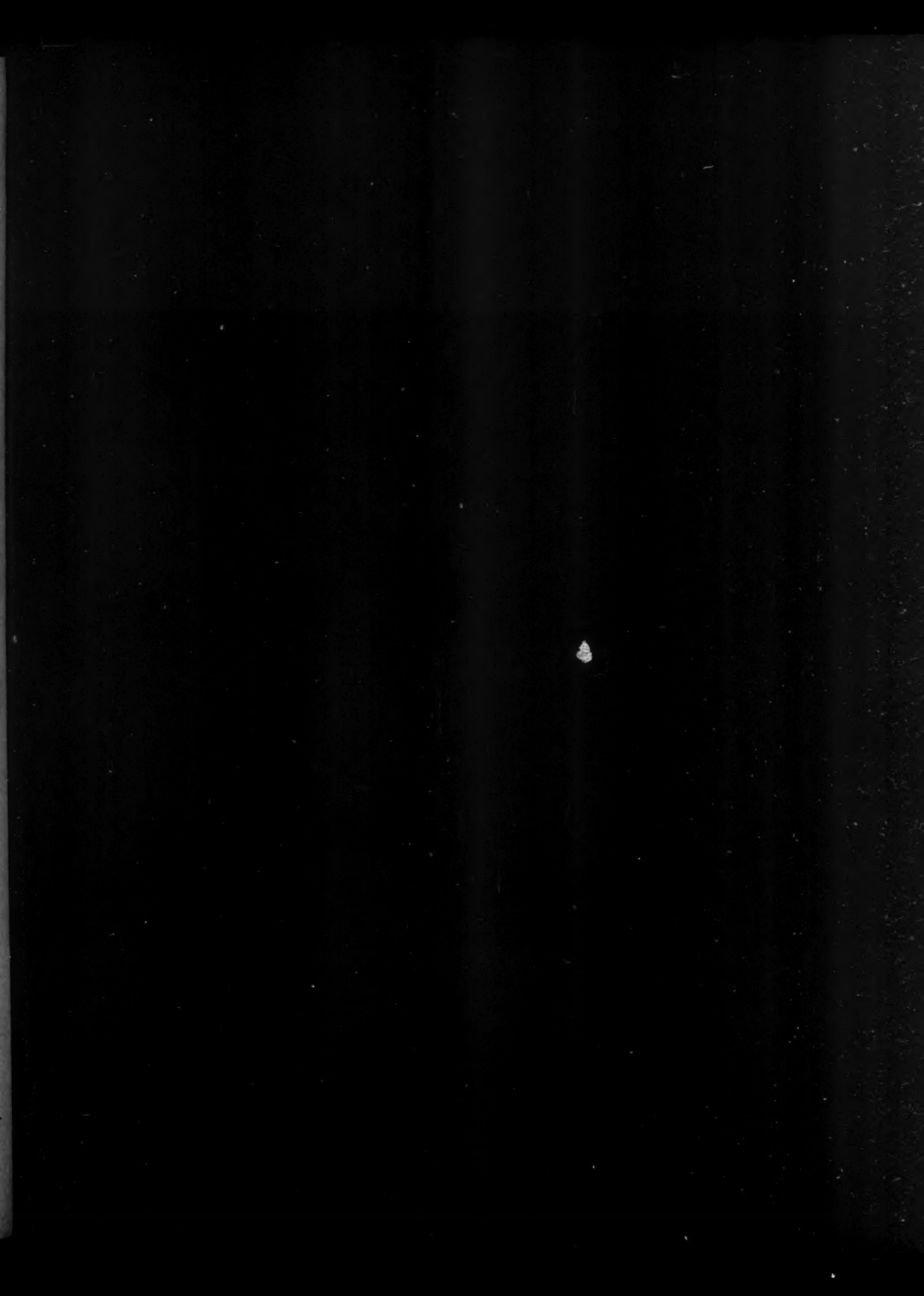
Editorial Office

MATHEMATICAL REVIEWS, Brown University, Providence 12, R. I.

Subscriptions: Price $13 per year ($6.50 per year to members of sponsoring societies). Checks should be made payable to MATHEMATICAL REVIEWS. Subscriptions should be addressed to MATHEMATICAL REVIEWS, Lancaster, Pennsylvania, or Brown University, Providence 12, Rhode Island.

This publication was made possible in part by funds granted by the Carnegie Corporation of New York, the Rockefeller Foundation, and the American Philosophical Society held at Philadelphia for Promoting Useful Knowledge. Its preparation is also supported currently under a contract with the Office of Naval Research, Department of the Navy, U.S.A. These organizations are not, however, the authors, owners, publishers, or proprietors of this publication, and are not to be understood as approving by virtue of their grants any of the statements made or views expressed therein.

Entered as second-class matter February 3, 1940 at the post office at Lancaster, Pennsylvania, under the act of March 3, 1879. Accepted for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in paragraph 4, section 538, P. L. and R. authorized November 9, 1940.



Mathematical Reviews

Vol. 10, No. 2 FEBRUARY, 1949 Pages 93-172

FOUNDATIONS

Słupecki, Jerzy. Les remarques sur la syllogistique d'Aristodte. Ann. Univ. Mariae Curie-Skłodowska. Sect. F. 1, 187–191 (1946). (Polish. French summary)

Łukasiewicz [Elements of Mathematical Logic, Warsaw, 1929] presented the following axiomatization of Aristotelian logic: *Uaa*, *Iaa*, *CKUmbUamUab*, *CKUmbImaIab*, where '*Uab*' means 'every a is b,' '*Iab*' means 'some a are b,' and the rule of substitution is limited to nonempty names. Słupecki presents another axiomatization where this limitation can be dropped: *CUabIab*, *CIabIba*, *CKUmbUamUab*, *CKUmbIamIab*. The first two of Łukasiewicz's axioms are independent of this last set. Słupecki also presents an interpretation of his axiomatisation which more or less conserves the classical meaning and permits substitution of empty names. This is done by interpreting it in the "ontology" of Leśniewski. The rules of interpretation are: $Uab = K\sum xexa\prod xCexaexb$, $Iab = \sum xKexaexb$. Słupecki's axioms follow from these rules alone without using any theorem of "ontology" based on the axiom for 'ϵ.' The sentence 'no a is b' is defined as: $Yab = NIab$, and 'some a are not b' as: $Oab = NUab$ (following Łukasiewicz).

H. Hiż (Cambridge, Mass.).

Łoś, Jerzy. Une preuve d'axiomatisation de la logique traditionelle. Ann. Univ. Mariae Curie-Skłodowska. Sect. F. 1, 211–228 (1946). (Polish. French summary)

Following the general ideas of Sleszyński [The Theory of Proof, v. 1, Kraków, 1925] the author analyzes traditional logic and especially laws of immediate inference by means of relations between extensions of arguments. Consider the expression '*exab*' (the common part of a and b is not empty). There are only 8 relations between the extensions of a and b ('E' represents equivalence, 'a'' class negation):

$$E\alpha abKKexabNexab'Nexa'b,\quad E\beta abKKexabNexab'exa'b,$$
$$E\gamma abKKexabexab'Nexa'b,\quad E\delta abKKexabexab'exa'b,$$
$$E\epsilon abKKNexabexab'exa'b,\quad E\zeta abKKNexabexab'Nexa'b,$$
$$E\eta abKKNxabNexab'exa'b,\quad E\theta abKKNexabNexab'Nexa'b.$$

The definitions of atomic sentences of classical logic read: $EU_3ab[\alpha, \beta, \eta, \theta]ab$, $EI_3ab[\alpha, \beta, \gamma, \delta]ab$, $EO_3ab[\gamma, \delta, \epsilon, \zeta]ab$, $EY_3ab[\epsilon, \zeta, \eta, \theta]ab$, where '$[z_1, \cdots, z_n]$' means the alternation of relations. In this interpretation not all formulas of classical logic are valid. The author presents the axiomatization for this limited system: CI_3abI_3ba, $CKU_3mbU_3amU_3ab$, $CKU_3mbI_3amI_3ab$. The functions of Słupecki's system [see the preceding review] are defined as follows: $[\alpha, \beta]$, $[\alpha, \beta, \gamma, \delta]$, $[\gamma, \delta, \epsilon, \zeta, \eta, \theta]$, $[\epsilon, \zeta, \eta, \theta]$. Consider the following set of sentences: (A_1) *Cexa''bexba*, (A_2) *Cexabexa''b*, (A_3) *CKNexmb'examexab*, (A_4) *Nexaa'*, (A_5) *Aexabexab'*. From A_1 and A_2 follow all laws of immediate inference. From A_1–A_2 follow also Sleszyński's and Słupecki's systems. These and Łukasiewicz's system follow from A_1–A_5.

H. Hiż (Cambridge, Mass.).

Łukasiewicz, Jan. The shortest axiom of the implicational calculus of propositions. Proc. Roy. Irish Acad. Sect. A. 52, 25–33 (1948).

By the "implicational analysis of propositions" the author means those formulas of the algebra of propositions which are formed from propositional variables by the implication connector (C) alone and are tautologies in the classical sense. He proves here that the formula *CCCpqrCCrpCCsp* (expressed in his notation) is sufficient, in combination with the usual rules of substitution and inference, as single axiom for that calculus. He gives further an outline of a proof that no shorter formula of that calculus can so serve as single axiom. [The first result was announced in Les entretiens de Zurich sur les fondements et la méthode des sciences mathématiques (1938), 1941, pp. 82–100; these Rev. 2, 338].

H. B. Curry (State College, Pa.).

Gerneth, Dal Charles. Generalization of Menger's result on the structure of logical formulas. Bull. Amer. Math. Soc. 54, 803–804 (1948).

Ein elementares Kriterium dafür, dass ein Ausdruck, der mit 2-stelligen Operatoren gebildet ist, eine Formel im Łukasiewicz-Kalkül ist, wird von 2 auf n verallgemeinert.

P. Lorenzen (Cambridge, England).

Mostowski, Andrzej. On absolute properties of relations. J. Symbolic Logic 12, 33–42 (1947).

Intuitively an absolute property of a relation relative to a given axiom set is one which that relation possesses in every model of the axiom set. More precisely, let R and S be two relations among real numbers, R a binary one and S a ternary one. Let A_0 be the field of the relations R and S and assume it infinite. Let A be a subset of A_0, $\mathfrak{A}$ a set of subsets of A, and α a set of subsets of $\mathfrak{A}$. The ordered triplet $<A, \mathfrak{A}, \alpha>$ will be denoted by 'M' and called a model. If A contains all elements of A_0, $\mathfrak{A}$ all subsets of A_0, and α all subsets of $\mathfrak{A}$, then M is an absolute model M_0 which is $<A_0, \mathfrak{A}_0, \alpha_0>$.

Let Λ_M mean that the model M fulfills a finite or denumerably infinite set Λ of axioms, i.e., that all the formulae of Λ yield truths if relativized to M. If P is some formula of the system, P_M is the relativization of P to M, i.e., roughly, the formula which results if we require that the variables of P range over the objects of M. Let Φ be a conjunction (finite or possibly even infinite) of properties $\hat{u}\hat{v}(u=R.v=S.P)$. Then Φ_M is the property Φ relativized to M. We say now that Φ is an absolute property with respect to Λ, if $(M)(\Lambda_M . \supset . [\Phi_{M_0}(R, S) = \Phi_M(R, S)])$ holds.

The author seeks a necessary condition for the absoluteness of Φ. Although he restricts himself to the two relations R and S, his results can presumably be extended to any number of relations. By means of the Cantor discontinuous

set C of real numbers $\sum_{i=1}^{\infty} c_i \cdot 3^{-i}$, where $c_i = 0$ or $c_i = 2$, he associates a real number to each binary or ternary relation of integers. A set of ordered couples of real numbers is introduced, the set (*) $E_{rs}[r - 4\varepsilon C . s - 8\varepsilon C . \Phi_N(R_r, R_s)]$, where R_r and R_s are respectively the relations assigned to the real numbers r and s, and N is $<N_0, \mathfrak{N}, \gamma>$ where N_0 is the set of all positive integers, $\mathfrak{N}$ the set of subsets of N_0, and γ the set of all subsets of $\mathfrak{N}$. Using several theorems on Borel sets, it is then shown that, if Φ is absolute relative to Λ, then the set (*) is Borelian and its class at most ω. It is presupposed that A_0 is infinite and that Λ includes the axiom of extensionality. Several further theorems are proved, including one which establishes that the relation of similarity (being of the same cardinality) is not absolute.

R. M. Martin (Philadelphia, Pa.).

Destouches-Février, Paulette. Logique de l'intuitionisme sans négation et logique de l'intuitionisme positif. C. R. Acad. Sci. Paris **226**, 38–39 (1948).

Un système d'axiomes pour une logique que l'auteur présume correspondre à la conception des mathématiques intuitionistes sans négation de Griss [Nederl. Akad. Wetensch. Verslagen, Afd. Natuurkunde **53**, 261–268 (1944); Nederl. Akad. Wetensch., Proc. **49**, 1127–1133 = Indagationes Math. **8**, 675–681 (1946); ces Rev. **7**, 405; **8**, 307]. Cette logique diffère de la logique intuitioniste ordinaire (1) par le manque de la négation, (2) par l'introduction de paires de propositions incomposables, dont il est interdit de former la conjonction. Elle est donc une logique de complémentarité. En définissant la contradiction par exemple par $0 =_d (1 = 2)$ on peut définir une négation: $\neg p =_d (p \to 0)$. Pour cette négation les règles principales de la logique intuitioniste sont valables.

A. Heyting (Amsterdam).

Dijkman, J. G. Recherche de la convergence négative dans les mathématiques intuitionistes. Nederl. Akad. Wetensch., Proc. **51**, 681–692 = Indagationes Math. **10**, 232–243 (1948).

L. E. J. Brouwer avait proposé la définition: (D_1) Une suite $\{a_n\}$ de nombres réels est convergente négativement vers la limite s, si pour chaque nombre positif ϵ il est impossible de trouver une suite $\{n_i\}$ d'entiers tels que $n_1 < n_2 < \cdots$ et que $|s - a_{n_i}| > \epsilon$ pour $i = 1, 2, \cdots$ [J. Reine Angew. Math. **154**, 1–7 (1924)]. Belinfante croyait avoir démontré le théorème: (S) Si les suites $\{a_n\}$ et $\{b_n\}$ sont convergentes négativement vers les limites a, respectivement b, alors $\{a_n + b_n\}$ est convergente négativement vers $a + b$ [S.-B. Preuss. Akad. Wiss. **1929**, 639–660]. Le rapporteur a remarqué que cette démonstration n'est pas exacte. C'est pourquoi l'auteur propose la définition: (D_2) La suite $\{a_n\}$ est convergente négativement au sens strict vers la limite a, si pour chaque nombre positif ϵ il est impossible qu'il n'existe pas un nombre $N(\epsilon)$ tel que $|a - a_n| < \epsilon$ pour tout $n > N$. La propriété (D_1) est une conséquence de (D_2); (D_2) est plus faible que la négation double de la convergence positive. La propriété (D_2) est équivalent à la conjunction de (D_1) avec le critère négatif de Cauchy: Pour chaque nombre positif ϵ il est impossible qu'il n'existe pas un nombre $N(\epsilon)$ tel que $|a_m - a_n| < \epsilon$ pour toutes les valeurs de m et $n > N$. Pour les suites convergentes (D_2) le théorème (S) est démontré ainsi que le théorème analogue pour les produits. Si une suite est convergente (D_2) vers a et vers b, alors $a = b$. Considérations analogues sur les suites non-oscillantes et sur la convergence négative multiple.

A. Heyting (Amsterdam).

Marković, Ž. Sur la formation des théories mathématiques. Hrvatsko Prirodoslovno Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. **1**, 49–64 (1946). (Croatian. French summary)

ALGEBRA

Hussain, Q. M. Structure of some incomplete block designs. Sankhyā **8**, 381–383 (1948).

Let $\phi(n)$ denote the number of blocks having n varieties in common with the first block. The author proves the identity

$$ij(b-1) - k(i+j-1)(r-1) + k(k-1)(\lambda-1) = \sum_{0}^{k} (i-n)(j-n)\phi(n).$$

In the case of a symmetrical design the left side reduces to $r(r-1)(i-\lambda)(j-\lambda)/\lambda$ and in the case of the residual design to $ij[r(r-1) - \lambda]/\lambda - (i+j)(r-1)(r-\lambda) + \lambda(r-\lambda)^2$. Specializing to $\lambda = 2$ the author proves five propositions about designs with $b = \frac{1}{2}(k+1)(k+2)$, $v = \frac{1}{2}k(k+1)$, $r = k+2$, $\lambda = 2$. He then proves the impossibility of the design $v = 15$, $b = 21$, $r = 7$, $k = 5$, $\lambda = 2$ by tactical enumeration, which is greatly facilitated by the use of his propositions.

H. B. Mann.

Hussain, Q. M. Alternative proof of the impossibility of the symmetrical design with $\lambda = 2$, $k = 7$. Sankhyā **8**, 384 (1948).

The author gives an alternative proof of the impossibility of the design $v = b = 22$, $r = k = 7$, $\lambda = 2$.

H. B. Mann.

Kerawala, S. M., and Hanafi, A. R. Table of monomial symmetric functions of weight 12 in terms of power-sums. Sankhyā **8**, 345–359 (1948).

The authors continue to weight 12 their tables [Proc. Nat. Acad. Sci. India. Sect. A. **11**, 56–63 (1941); **12**, 81–96 (1942); these Rev. **8**, 191] expressing monomial symmetric functions in terms of power sums, a form of interest in statistics because of the connections of the latter with moments.

J. Riordan (New York, N. Y.).

Bacon, H. M. A matrix arising in correlation theory. Ann. Math. Statistics **19**, 422–424 (1948).

Defining $r_{ij} = 1 - |i-j|\rho$, $0 < \rho \leqq 2/(n-1)$, explicit formulas, obtained by elementary calculations, are given for the determinant $R_n = \|r_{ij}\|$, the cofactors R_{ij}, and the corresponding frequency function.

H. Wold (Uppsala).

Szarski, J., et Ważewski, T. Sur la relation entre le module d'un déterminant complexe et son déterminant réel, associé. Application à la théorie des formes hermitiennes et à celle des modules des matrices complexes. Ann. Soc. Polon. Math. **20** (1947), 1–6 (1948).

If $M = A + iB$ is a complex matrix, the correspondence

$$M \leftrightarrow M^* = \begin{bmatrix} A & -B \\ B & A \end{bmatrix}$$

is an isomorphism. From the identity

$$\det(M^*-\lambda I)=\det(M-\lambda I)\cdot\det(\bar{M}-\lambda I)$$

it follows that $|\det M^*|=|\det M|^2$ and that the characteristic roots of M^* are those of M and their conjugates. If M is Hermitian, M^* is symmetric. If $\sum h_{jk}x_j\bar{x}_k \leqq \sum h'_{jk}x_j\bar{x}_k$, both forms being Hermitian with respective characteristic roots $\lambda_1 \leqq \cdots \leqq \lambda_n$, $\lambda_1' \leqq \cdots \leqq \lambda_n'$, it is shown that $\lambda_i \leqq \lambda_i'$.

C. C. MacDuffee (Madison, Wis.).

Foulkes, H. O. Rational solutions of the matrix equation $XA=BX$. Proc. London Math. Soc. (2) **50**, 196–209 (1948).

A method is described for finding a complete set of solutions of $XA=BX$ with elements in any field containing the elements of A and B. If A and B satisfy the same algebraic equation, some solutions can be obtained by an alternative procedure. *C. C. MacDuffee* (Madison, Wis.).

Colombo, Xenia. Sulle funzioni delle matrici. Pont. Acad. Sci. Acta **6**, 375–386 (1942).

Using the definition of function of a matrix due to Fantappiè [C. R. Acad. Sci. Paris **186**, 619–621 (1928)], the author obtains an explicit form for Φ^n, where Φ is a matrix of order 2 or 3. *C. C. MacDuffee.*

Lee, H. C. Canonical factorization of pseudo-unitary matrices. Proc. London Math. Soc. (2) **50**, 230–241 (1948).

A pseudo-unitary matrix A is one satisfying $A^*GA=G$, where $G=I_p+(-I_q)$ and the star denotes the conjugate transpose. Writing

$$A=\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix},$$

where A_1 is $p\times p$, the author shows that the characteristic values of $A_1A_1^*$ and $A_4A_4^*$ are all greater than or equal to 1 and those greater than 1 are the same for the two matrices, say $\cosh^2\theta_i$ $(i=1,\cdots,r)$, with the "splitting invariants" $\theta_i>0$. In terms of the distinct θ_i, A is factored into $A=(U+V)N(X+Y)$, where U and X are $p\times p$ and unitary, V and Y are $q\times q$ and unitary, and the "nucleus" N has the form

$$N=I_{p-r}+\begin{pmatrix} T & S \\ S & T \end{pmatrix}+I_{q-r}$$

with

$$T=\cosh\theta_1 I_{r_1}+\cdots+\cosh\theta_s I_{r_s},$$
$$S=\sinh\theta_1 I_{r_1}+\cdots+\sinh\theta_s I_{r_s},$$

$\theta_1<\cdots<\theta_s$ and $r_1+\cdots+r_s=r\leqq\min(p,q)$. Leaving N unchanged it is possible to require U and V to have certain elements zero and others real and nonnegative. In case the latter are all different from zero, the requirements are shown to lead to unique U, V, X and Y. Using the nucleus factorization it is shown that, in the real case, A, if of determinant plus one, can be written as a product of $\frac{1}{2}(p+q)(p+q-1)-(p-r)(q-r)-\sum_{i=1}^{s}\frac{1}{2}r_i(r_i-1)-r$ plane rotations and r pseudo-plane rotations. *W. Givens.*

Reicheneder, K. Der Indizes-Kalkul und seine Anwendung in der Algebra. Bull. École Polytech. Jassy [Bul. Politehn. Gh. Asachi. Iaşi] **3**, 48–60 (1948).

Finding the concepts of matric algebra too abstract for his liking, the author suggests an alternative notation which is essentially that of tensor analysis. He solves a number of problems, mostly in geodesy, by his "matrikels."

C. C. MacDuffee (Madison, Wis.).

Fumi, F. Operatori matriciali di simmetria e quaternioni. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **3**, 573–580 (1947).

Die orthogonalen Transformationen des R_3 werden sowohl in Quaternionen wie auch in Matrizen angegeben, indem die allgemeine Bewegung in ihre rotativen und translativen Bestandteile zerlegt wird. *J. J. Burckhardt* (Zürich).

Jacobsthal, Ernst. Sur les nombres hypercomplexes. Norske Vid. Selsk. Forh., Trondhjem **20**, no. 8, 29–32 (1947).

The author considers a real linear associative algebra in 3 units which multiply like the cube roots of unity. His "measure" is simply the absolute value of the determinant of the first regular matric representative of the vector. He shows that vectors of measure 0 are divisors of zero, others have inverses. *C. C. MacDuffee* (Madison, Wis.).

Châtelet, François. Hyperquadriques dans un corps arbitraire. C. R. Acad. Sci. Paris **226**, 1578–1580 (1948).

Extension to quadratic forms in an arbitrary number of variables of the ideas of an earlier paper [same vol., 1233–1235 (1948); these Rev. **9**, 560]. No proofs are given.

D. B. Scott (London).

Todd, J. A. Combinant forms associated with a pencil of conics. Proc. London Math. Soc. (2) **50**, 150–168 (1948).

The author determines the complete irreducible system of concomitants of the form $A_x^2a_y$, quadratic in the ternary variable $x=(x_0,x_1,x_2)$ and linear in the binary variable $y=(y_0,y_1)$. Since the equation $A_x^2a_y=0$ may be interpreted geometrically as a pencil of conics, the system thus determined is a complete system of combinants of a pencil of conics.

The author shows that there is a system M of 11 forms belonging to the complete system of $A_x^2a_y=S_0y_0+S_1y_1$ and that the set of coefficients of the power products of y_0 and y_1 in these forms is equivalent to the known complete system of the two ternary quadratics S_0 and S_1. Of these 11 forms, five are independent of y, three are linear, two are quadratic and one is a cubic in y. A complete system for three linear forms, two quadratic forms and one linear form is found. This system contains 216 forms. These 216 forms, when the general coefficients are replaced by the corresponding coefficients of the forms of M, together with the five forms in M independent of y, contain the required system of combinants.

In reducing the 221 forms to 65 the author takes the two quadratics S_0 and S_1 in a special canonical form and calculates the 65 forms explicitly. He also exhibits the syzygies by means of which the remaining 156 forms may be expressed in terms of the 65. He concludes by interpreting some of the simpler forms geometrically.

J. Williamson (Flushing, N. Y.).

Abstract Algebra

Krishnan, V. S. Extensions of partially ordered sets. I. General theory. J. Indian Math. Soc. (N.S.) **11**, 49–68 (1947).

The author observes that the definition of extension by MacNeille [Trans. Amer. Math. Soc. **42**, 416–460 (1937)] does not say what MacNeille must have intended it to say

and must be modified. Much of MacNeille's work is not affected by this change. However, his theorem 9.11 on extensions of extensions is trivial as it stands, and false if modified in the obvious manner, as the author shows in 3.6, so it must be weakened. Likewise the author's example 4 disproves MacNeille's side remarks (following the proof of theorem 9.8) about minimality in the case of a superset instead of extension. [On the other hand, the author's example 2 is invalid since $f[I]$ is not always an ideal and then is not in L as claimed.] The author concludes with the beginning of a modified theory. *P. M. Whitman.*

Duschek, A. Über eine neue Art von algebraischen Bereichen. Monatsh. Math. **52**, 89–123 (1948).

The author considers algebraic systems with two operations, addition and multiplication, satisfying the commutative and associative laws but no law interrelating the two. It is required further that the conditions $2A=2B$, $A^2=B^2$, shall each imply $A=B$. Such a system is called a "class." One may also make the stronger requirement that any of the relations $kA=kB$, $A^k=B^k$ (for some k) shall have the same implication. It is shown that if in a class $k\leqq n$ are the smallest integers such that $kA=nA$ then $k=1$ and n is even. A "class" B_2 with two elements is a Boolean algebra denoted by $(0, \infty)$. A special "class" of three elements can be interpreted in three-valued logic. The author studies polynomials with coefficients and variable values in B_2 and reduces them to normal forms. The main application is to electrical networks. It is shown how networks with certain prescribed switching properties may be constructed in their simplest form by means of the normalized polynomials.

O. Ore (New Haven, Conn.).

***McCoy, Neal H. Rings and Ideals.** Carus Monograph Series, no. 8. The Open Court Publishing Company, LaSalle, Ill., 1948. xii+216 pp. $3.00 (Members of the Mathematical Association of America may purchase one copy at $1.75, order to be placed with the Secretary-Treasurer).

The author has set himself the task of presenting an introduction to that branch of abstract algebra dealing with the theory of rings. To this end he has given an exposition of the basic definitions and theorems concerning rings [chapters I to IV], as well as of the more special aspects of the theory, such as commutative rings, direct sums and matrix rings. Much of the material of this monograph is here made available for the first time in book form. A summary by chapters follows. (I) Definitions and fundamental properties. (II) Polynomial rings. The treatment here is more general than the usual one in that some of the results are carried through without the assumption that the coefficient ring is commutative. (III) Ideals and homomorphisms. (IV) Some imbedding theorems. The usual constructions are given for adjoining to a field a root of a polynomial, for adjoining a unit element, and for the formation of the quotient ring. (V) Prime ideals in commutative rings. An exposition of Krull's theory of ideals in rings without chain condition, hitherto available in full only in Krull's original papers. (VI) Direct and subdirect sums. Included are criteria, in terms of the vanishing of the radical of the zero ideal (in the sense of commutative ideal theory) and of the Jacobson radical, for a commutative ring to be a subdirect sum of, respectively, integral domains and fields. (VII) Boolean rings and some generalizations. Included is a simple proof of Stone's representation theorem. (VIII) Rings of matrices. The treatment is novel in that the elements of the matrices are taken from an arbitrary commutative ring, possibly having zero divisors. (IX) Further theory of ideals in commutative rings. This chapter discusses the representation of ideals as intersections of primary ideals. There follows a discussion of Noetherian rings and of the representation of algebraic manifolds by ideals.

As can be seen from the above summary, various parts of ring theory are discussed, but one feels that in some cases the author has not gone deeply enough into a particular theory to show its significance or indicate its range of application. The reviewer regrets, for example, that there is no treatment of Wedderburn's structure theorems, nor even any reference to them in the bibliography. At least the theorem on the decomposition of a semi-simple ring would have fitted naturally into chapter VI. Similarly, the theory of factorization in algebraic number and function fields, in Noether's abstract formulation, could appropriately have been included in the last chapter and, in the reviewer's opinion, would be of greater interest to the nonspecialist than Krull's ideal theory of non-Noetherian rings.

The book presupposes no previous training in algebra (except that in certain parts determinants are used). Because of the clarity with which it is written and because of the consistent use of simple examples to motivate the introduction of new ideas, it can be read without difficulty by those who are new to the subject. Of course, as the author remarks, a certain amount of mathematical maturity is necessary to understand the abstract approach. For those interested in additional study, references for further reading are given at the end of each chapter to a bibliography collected at the end of the book. *I. S. Cohen.*

Rickart, C. E. One-to-one mappings of rings and lattices. Bull. Amer. Math. Soc. **54**, 758–764 (1948).

Soient R et S deux anneaux, f une application biunivoque de R sur S, telle que $f(xy)=f(x)f(y)$. L'auteur montre que f est nécessairement un isomorphisme de R sur S (c'est-à-dire qu'on a aussi $f(x+y)=f(x)+f(y)$) dans les deux cas suivants: (1) R est un anneau booléien; (2) R est un anneau contenant une famille J d'idéaux minimaux à droite avec les propriétés suivantes: (a) il n'y a aucun élément $x\neq 0$ de R qui annule à droite tous les idéaux de J; (b) chaque idéal de la famille J est de dimension au moins 2 sur son corps d'endomorphismes (R-module à droite). Ce second cas est en particulier celui de l'anneau des endomorphismes d'un espace de Banach de dimension au moins 2. [Note du rapporteur: c'est aussi le cas de tout anneau d'endomorphismes d'un espace vectoriel de dimension au moins 2 sur un corps quelconque, lorsque cet anneau contient tous les endomorphismes de rang fini.] L'auteur démontre enfin un théorème tout à fait analogue pour deux lattices distributifs L, M et une application biunivoque f de L sur M telle que $f(x\cap y)=f(x)\cap f(y)$: on a alors nécessairement aussi $f(x\cup y)=f(x)\cup f(y)$. *J. Dieudonné* (Nancy).

Szele, T. Ein Satz über die Struktur der endlichen Ringe. Acta Univ. Szeged. Sect. Sci. Math. **11**, 246–250 (1948).

K. Shoda [Math. Ann. **100**, 674–686 (1928)] showed that the endomorphism ring of a finite p-group of type $(\alpha_1, \cdots, \alpha_n)$ is isomorphic with the ring $\mathfrak{M}_p(\alpha_1, \cdots, \alpha_n)$ of all n-by-n integral matrices (a_{ik}) such that a_{ik} for $i>k$ is divisible by $p^{\alpha_k-\alpha_i}$. The author applies this to the additive group $\mathfrak{R}^+$ of any finite ring $\mathfrak{R}$ of prime power order, noting

that $X \to XA$ is an endomorphism of $\mathfrak{R}^+$. If $XA=0$ for all X implies $A=0$, then $\mathfrak{R}$ is isomorphic with a subring of $\mathfrak{M}_p(\alpha_1, \cdots, \alpha_n)$. Otherwise one first adjoins an identity element to $\mathfrak{R}$ and obtains $\mathfrak{R}$ isomorphic with a subring of $\mathfrak{M}_p(\alpha_1, \alpha_1, \alpha_2, \cdots, \alpha_n)$. *A. H. Clifford* (Baltimore, Md.).

Uzkov, A. I. On rings of quotients of commutative rings. Mat. Sbornik N.S. 22(64), 439–441 (1948). (Russian)

Chevalley [Bull. Amer. Math. Soc. 50, 93–97 (1944); these Rev. 5, 226] has extended the usual definition of the ring of quotients of a commutative ring R with respect to a multiplicatively closed subsystem S without divisors of zero [cf. Grell, Math. Ann. 97, 490–523 (1927), in particular, p. 499; or van der Waerden, Moderne Algebra, v. 1, 2d ed., Springer, Berlin, 1937, p. 45, "Aufgabe"] to the case where divisors of zero occur in S. Chevalley's construction assumes the maximal condition for ideals to hold in R. The author shows that by a suitable definition of the ring of quotients it is possible to get rid of this restriction and to construct the ring of quotients under very general conditions.

Let R be a commutative ring and S a set of elements of R, closed under multiplication and not containing 0. A homomorphic mapping χ of R onto some other ring R' with unit element is called S-reducing if the elements of the set $\chi(S)$ have inverses in R'. A ring R_S is called ring of quotients of R with respect to S, if there exists a homomorphic mapping χ of R onto R_S with the following properties: (1) χ is an S-reducing mapping of R onto R_S; (2) χ is not an S-reducing mapping of R onto any proper subring of R_S; (3) if R' is an arbitrary ring and χ' an S-reducing mapping of R onto R', then there exists a homomorphic mapping ψ of R_S onto R' such that $\chi'(r)=\psi[\chi(r)]$ for any $r \in R$. For any S-reducing homomorphism χ the kernel N_χ satisfies $N_\chi \supseteq N_S$, where N_S is the ideal, in R, of those elements x for which $xs=0$ for some $s \in S$.

A ring of quotients can now be constructed in the usual manner, i.e., with pairs of elements (r, s), where $r \in R$, $s \in S$. The equivalence relation between such pairs, however, has to be defined as: (r_1, s_1) equivalent to (r_2, s_2) if there exists an element $s \in S$ such that $r_1 s_2 s = r_2 s_1 s$. It is then shown that the set of classes of equivalent pairs forms a ring with the obvious definitions of addition and multiplication. The natural mapping $\chi(r)=(rs, s)$ of R onto R_S is then a homomorphism which satisfies the condition (1)–(3) above. [For (3) choose the mapping $\psi(r, s)=\chi'(r)/\chi'(s)$; the kernel of this mapping is N_S.] This ring of quotients R_S is unique up to isomorphism.

To each ideal I of R there corresponds uniquely an ideal $\chi(I) \cdot R_S$, the "extension" of I in R_S, which is denoted (following Chevalley) by IR_S; conversely to every ideal I_S in R_S there corresponds an ideal $\chi^{-1}[I_S \cap \chi(R)]$ in R, the "base" of I_S in R, which is denoted by $I_S \cap R$ (these ideals correspond to Grell's "Erweiterungsideal" and "Verengungsideal"). The following properties are established: (1) $(I_S \cap R) \cdot R_S = I_S$; (2) for any two I_{1S} and I_{2S}, $(I_{1S} \cap I_{2S}) \cap R = (I_{1S} \cap R) \cap (I_{2S} \cap R)$ (note that the symbol $\cap$ occurs with two meanings: set-theoretical intersection and base ideal as defined above; this rule justifies the choice of the "base" ideal); (3) if R is a prime ideal in R and Q a primary ideal belonging to P then either $PR_S = QR_S = R_S$ or PR_S is prime and QR_S a primary ideal belonging to PR_S, and in the latter case $PR_S \cap R = P$ and $QR_S \cap R = Q$; (4) if $I = \cap_{i=1}^{l} Q_i$ is an irredundant intersection of primary ideals, then $IR_S = \cap_{i=1}^{l} Q_i R_S$ and this is again irredundant if the components $Q_i R_S = R_S$ (i.e., those for which $P_i \cap S$ is not empty) are excluded. Finally it is shown that if the ring R satisfies the maximal condition for ideals the author's definition of the ring of quotients coincides with that given by Chevalley. *K. A. Hirsch* (Newcastle-upon-Tyne).

Geymonat, Ludovico. Un teorema per le aritmetiche generalizzate. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. 81–82, 59–62 (1948).

Geymonat, Ludovico. Ancora su di un teorema per le aritmetiche generalizzate. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. 81–82, 63–66 (1948).

It is shown that a ring with a positive norm and Euclidean algorithm has unique factorisation. This result is known. *O. Todd-Taussky* (London).

Albert, A. A. On the power-associativity of rings. Summa Brasil. Math. 2, no. 2, 21–32 (1948).

This paper is concerned with the proof of the following theorem. Let A be a (nonassociative) ring in which the identities $x \cdot x^3 = x^3 \cdot x$, $x^2 \cdot x^2 = (x^2 \cdot x) \cdot x$ hold. If either (i) A is of characteristic zero, or (ii) A is commutative and has characteristic prime to 30, every element x of A generates an associative subring of A. The author remarks that, to prove this theorem, it is both necessary and sufficient to show that the identity $x^\lambda \cdot x^\mu = x^{\lambda+\mu}$ holds for all x in A and all pairs of positive integers λ, μ. The symbol x^λ occurring in this identity is defined inductively by $x^1 = x$, $x^{\lambda+1} = x^\lambda \cdot x$.

The stages of the proof are as follows. The two identities $x \cdot x^3 = x^2 \cdot x$ and $x^2 \cdot x^2 = (x^2 \cdot x) \cdot x$ are polarised to yield two identities involving, respectively, x, y, z and x, y, z, w, where x, y, z, w are arbitrary elements of A. This stage involves the assumption that A has characteristic prime to 2. From these identities, it follows that $x^\lambda \cdot x^\mu = x^{\lambda+\mu}$ if $\lambda+\mu \leqq 4$. Next, after considerable computation, the following lemma is proved. If A has characteristic prime to 30, and $x^\lambda \cdot x^\mu = x^{\lambda+\mu}$ whenever $\lambda+\mu < n$, where $n \geqq 5$, then

$$2x^{n-\alpha} \cdot x^\alpha = 2 \cdot x^{n-1} \cdot x + (\alpha-1)(x^{n-1} \cdot x - x \cdot x^{n-1}), \qquad \alpha = 1, \cdots, n-1.$$

From this result, the two parts of the main theorem follow almost immediately.

The paper closes with counter-examples to show that the conditions imposed on the characteristic cannot be weakened without invalidating the theorem. [The formula at the foot of page 5 should read

$$0 = C(x+\mu z, y) = C(x, y) + \mu F(x, y, z) + \mu^2 C(z, y) = \mu F(x, y, z)$$

and the first three lines at the top of page 6 are consequently unnecessary, as $F(x, y, z)=0$ follows immediately.]

D. Rees (Manchester).

Šafarevič, I. R. Investigations on the theory of finite extensions. Uspehi Matem. Nauk (N.S.) 2, no. 2(18), 223–226 (1947). (Russian)

[Summary of a thesis at the V.A. Steklov Mathematical Institute, 1946.] The introduction consists of a brief survey over the development of the theory of algebraic equations from Cardano and Ferrari via Abel and Galois to the present time, and a short account of three cases where a complete enumeration of all extensions of a field k with a commutative group can be given ((1) field of rational functions of one complex variable: theory of Abelian functions; (2) field of rational numbers: theory of class fields; (3) field of p-adic numbers: local theory of class-fields).

The dissertation falls into three parts. In the first part finite extensions with a p-group as Galois group ("p-extensions") of the field of p-adic numbers or a finite extension of it are studied. The ground field must not contain the pth roots of unity (other than 1). If k is such an extension of degree n_0, and if γ denotes the free group with n_0+1 generators, then a one-to-one correspondence between these extensions and the self-conjugate subgroups of γ with prime-power index is established. If K_1 and N_1 are coordinated, the Galois group of K_1 is isomorphic to γ/N_1; and if K_2 and N_2 are also coordinated, and $K_1 \supset K_2$, then $N_1 \supset N_2$ and vice versa. In this manner the problem of extensions with prescribed group is solved: if the p-group G of order p^n and rank (minimal number of generators) α is given, then over the field K there exist extensions with G as Galois group if and only if $\alpha \leqq n_0+1$, and the number of distinct extensions is $\alpha^{-1}p^{(n_0+1)(n-\alpha)}(p^{n_0+1}-1)(p^{n_0+1}-p)\cdots(p^{n_0+1}-p^{\alpha})$.

In the second part unramified p-extensions of the field of algebraic functions of characteristic p are studied. Known results in the case of a commutative Galois group are extended to the case of a noncommutative Galois group, again by means of the self-conjugate subgroups of some free group. The arithmetical character of the problem complicates the demonstrations appreciably.

The last part deals with a problem in the theory of p-adic fields. Let K be an Abelian extension of a field of p-adic numbers k, given by the multiplicative group H of the elements of k; let now k be an extension of some subfield R. The Galois group of K over R is constructed with the help of the theory of algebras. No proofs of the results are given.

K. A. Hirsch (Newcastle-upon-Tyne).

Barsotti, Iacopo. Algebre senza base finita. I. Ann. Mat. Pura Appl. (4) **26**, 57–66 (1947).

In this paper [extracted from the author's thesis] the author lays the foundation of a theory of linear spaces and algebras of infinite order. Particular attention is paid to what can be proved without the axiom of choice.

I. Kaplansky (Princeton, N. J.).

Lombardo-Radice, L. Sulle condizioni di appartenenza al radicale per gli elementi di un'algebra legata a un gruppo finito. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **4**, 53–54 (1948).

Remarks on the radical of a modular group algebra.

I. Kaplansky (Princeton, N. J.).

Thrall, R. M. Some generalizations of quasi-Frobenius algebras. Trans. Amer. Math. Soc. **64**, 173–183 (1948).

An algebra $\mathfrak{A}$ with unit element over a field $\mathfrak{k}$ is called a Frobenius algebra if its right and left regular representations, R and L, are equivalent; it is called a quasi-Frobenius algebra (QF) if every indecomposable direct constituent of R is equivalent to some indecomposable direct constituent of L. The study of QF algebras was initiated by Nakayama [Ann. of Math. (2) **40**, 611–633 (1939); **42**, 1–21 (1941); Proc. Imp. Acad. Tokyo **16**, 285–289 (1940); **17**, 53–56 (1941); these Rev. **1**, 3; **2**, 344, 245, 344] and has been carried further by Nesbitt and Thrall [Ann. of Math. (2) **47**, 551–567 (1946); these Rev. **8**, 64]. A knowledge of the results of this latter is assumed throughout the present paper.

Every QF algebra $\mathfrak{A}$ has the following properties: (1) every faithful representation of $\mathfrak{A}$ is its own second commutator; (2) every primitive left (right) ideal has a unique minimal subideal; (3) $\mathfrak{A}$ has a unique minimal faithful representation (so that the deletion of any direct constituent from it would not leave a faithful representation). These properties, however, are not characteristic of QF algebras but hold in wider classes and in the present paper the author investigates the interdependence of algebras with these properties and QF algebras. Indices are used to denote those of the above properties which a $\mathfrak{k}$-algebra $\mathfrak{A}$ with unit element is assumed to possess. The author obtains the following results: The class QF-1 properly contains the class QF. (Whether QF-1 contains QF-2 or QF-3 remains an open problem.) The class QF-3 properly contains the class QF-2. Necessary and sufficient conditions for a QF-2 and a QF-3 algebra to be QF are derived. *K. A. Hirsch* (Newcastle-upon-Tyne).

THEORY OF GROUPS

Dyubyuk, P. E. On the number of subgroups of an Abelian p-group. Izvestiya Akad. Nauk SSSR. Ser. Mat. **12**, 351–378 (1948). (Russian)

The author proves several results about the number of subgroups of a finite Abelian p-group, generalizing theorems of Miller and of P. Hall. Let $\varphi_{\alpha,\beta}$ be the number of subgroups of order p^{β} of an elementary Abelian p-group of order p^{α}. For an (additive) Abelian p-group $\mathfrak{P}$, let invariants d_{i+1} be the rank of $p^i\mathfrak{P}$, $i=0, 1, 2, \cdots$, where the rank of (0) is taken as 0. The number of subgroups of $\mathfrak{P}$ having an arbitrary set of such invariants $a_1, \cdots, a_k$ (further a's being 0) is computed, using identities developed earlier for the φ's. Let $H(\alpha)$ [the author uses a more detailed symbol] be the number of subgroups of $\mathfrak{P}$ of order (alternately, index) p^{α}. A number of congruences and equalities involving $H(\alpha)$ are established, generalizing results of Miller [Bull. Amer. Math. Soc. **26**, 66–72 (1919)] and of P. Hall [Proc. London Math. Soc. (2) **36**, 29–95 (1933)]. Typical of these are $H(\alpha)=H_{\alpha}(\alpha)$, where the latter is the $H(\alpha)$ computed for a p-group with the invariants $d_1, \cdots, d_{\alpha}$; $H(\alpha) \equiv \varphi_{s,\alpha'}$ (mod p^s), where α is suitably restricted and α' and s are appropriately defined; and $H(\alpha) \equiv H(d_1+d_2, d_3, d_4, \cdots; \alpha)$ (mod p^{d_1}), the right member being the $H(\alpha)$ of a p-group with the invariants $d_1+d_2, d_3, d_4, \cdots$. Let $\mathfrak{P}$ be of rank d and of order p^n. It is proved that the number Φ_{α} of subgroups of $\mathfrak{P}$ of order (index) p^{α} is congruent to $\varphi_{\alpha,\alpha}$ (mod p^{δ}), generalizing a result of P. Hall [Proc. London Math. Soc. (2) **40**, 468–501 (1936)]. In many cases, $\Phi_{\alpha} \equiv 1+p+2p^2$ (mod p^3).

F. Haimo (St. Louis, Mo.).

Chow, Wei-Liang. On the algebraical braid group. Ann. of Math. (2) **49**, 654–658 (1948).

The paper is related to the work of A. Markoff [Foundations of the algebraic theory of tresses, Trav. Inst. Math. Stekloff **16** (1945); these Rev. **8**, 131], Artin [same Ann. (2) **48**, 101–126 (1947); these Rev. **8**, 367] and Bohnenblust [same Ann. (2) **48**, 127–136 (1947); these Rev. **8**, 367]. By a suitable choice of generators in the algebraic braid group the author arrives at a normal form, thereby giving an alternative solution of the word problem in this group. This normal form is used (a) to prove (using results of Artin) that the algebraic braid group and the geometric one are isomorphic, (b) to find the center of the algebraic braid group.

S. Eilenberg (New York, N. Y.).

Hall, Marshall, Jr., and Radó, Tibor. On Schreier systems in free groups. Trans. Amer. Math. Soc. **64**, 386–408 (1948).

Let $\mathfrak{F}$ be the free group over a countable set of generators S_i. A Schreier system is a subset $\mathfrak{G}$ of $\mathfrak{F}$ containing the

empty word 1 of $\mathfrak{F}$ and such that if $S_{i_1}^{\epsilon_1}\cdots S_{i_r}^{\epsilon_r}$ (each $\epsilon=\pm 1$) is the reduced expression of an element G of $\mathfrak{G}$, then $\mathfrak{G}$ contains each beginning section $S_{i_1}^{\epsilon_1}\cdots S_{i_q}^{\epsilon_q}$ $(q=1, 2, \cdots, r-1)$ thereof. In what follows, S always means a generator of $\mathfrak{F}$, G an element of a Schreier system $\mathfrak{G}$, and $\epsilon=\pm 1$. Let $\mathfrak{U}$ be any subgroup of $\mathfrak{F}$. O. Schreier [Abh. Math. Sem. Hamburg. Univ. 5, 161–183 (1927)] showed that it is always possible to select representatives for the left cosets of $\mathfrak{F}$ relative to $\mathfrak{U}$ such that the set $\mathfrak{G}$ thereof is a Schreier system, and in fact such that $\mathfrak{G}$ can be chosen to be "shortest" in the sense that no element in $\mathfrak{U}G$ has shorter length than G. For each $F\in\mathfrak{F}$ let $\Psi(F)$ denote the element of $\mathfrak{G}$ representing the left coset $\mathfrak{U}F$. Schreier [loc. cit.] showed that the set of all distinct elements $U\neq 1$ of $\mathfrak{F}$ of the form $GS[\Psi(GS)]^{-1}$ constitute a set of free generators of $\mathfrak{U}$ (and hence that every subgroup of a free group is free).

A Schreier system $\mathfrak{G}$ and a subgroup $\mathfrak{U}$ of $\mathfrak{F}$ are said to be associated if each left coset of $\mathfrak{F}$ relative to $\mathfrak{U}$ contains exactly one element of $\mathfrak{G}$. By Schreier's results, there is a Schreier system $\mathfrak{G}$ associated with any given subgroup $\mathfrak{U}$ of $\mathfrak{F}$. Conversely, the authors prove that to a given Schreier system $\mathfrak{G}$ there exists an associated subgroup $\mathfrak{U}$ of $\mathfrak{F}$ if and only if $\mathfrak{G}$ satisfies the following condition (C): for each S the number of elements of $\mathfrak{G}S$ not in $\mathfrak{G}$ equals the number of elements of $\mathfrak{G}S^{-1}$ not in $\mathfrak{G}$, i.e., both numbers are finite and equal, or both are (countably) infinite.

A function $\varphi(H)$ defined for all elements H of $\mathfrak{F}$ of the form $H=GS^{\epsilon}$ is termed admissible with respect to $\mathfrak{G}$ if: (a) $\varphi(H)\in\mathfrak{G}$ for all H; (b) if $H\in\mathfrak{G}$, say $H=GS^{\epsilon}=G'$, then $\varphi(H)=G'$; (c) $\varphi[\varphi(GS^{\epsilon})S^{-\epsilon}]=G$ for all G, S, ϵ. For example, the representative-function Ψ (for a $\mathfrak{U}$ associated with $\mathfrak{G}$) is admissible with respect to $\mathfrak{G}$. For given $\mathfrak{G}$, one obtains all possible admissible functions $\varphi(H)$ as follows. For each S separately, first satisfy (b) with $\epsilon=+1$. Then for all remaining G define $\varphi(GS)=G'$ where $GS\leftrightarrow G'S^{-1}$ is any one-to-one correspondence between $\mathfrak{G}S-\mathfrak{G}$ and $\mathfrak{G}S^{-1}-\mathfrak{G}$ (condition C). Then $\varphi(GS^{-1})$ is determined by (c). For any such φ, the set of distinct elements $U\neq 1$ of $\mathfrak{F}$ of the form $GS[\varphi(GS)]^{-1}$ are free generators of a subgroup $\mathfrak{U}$ of $\mathfrak{F}$; $\mathfrak{U}$ is associated with $\mathfrak{G}$, and $\Psi(H)=\varphi(H)$ for all $H=GS^{\epsilon}$.

Now assume that the number of generators S of $\mathfrak{F}$ is finite. Then we can order the generators U of $\mathfrak{U}$ by length: $l(U_i)\leqq l(U_j)$ if $i<j$. The sequence $\{U_i\}$ is called a minimal system of generators of $\mathfrak{U}$ if $l(U_i)\leqq l(U_i')$ $(i=1, 2, \cdots)$ for every other sequence $\{U_i'\}$ of generators of $\mathfrak{U}$. If $\mathfrak{G}$ is a shortest Schreier system associated with any subgroup $\mathfrak{U}$ of $\mathfrak{F}$, then the corresponding system $\{U_i\}$ of free generators of $\mathfrak{U}$ is minimal. *A. H. Clifford* (Baltimore, Md.).

Boerner, Hermann. Über die rationalen Darstellungen der allgemeinen linearen Gruppe. Arch. Math. 1, 52–55 (1948).

It is well known that the irreducible representations of the general linear group (in the space of tensors) are obtained by imposing certain symmetry conditions on the tensors. The author now sketches the proof of a theorem which makes it possible actually to find a base in each irreducible subspace of tensor space and so to write down the matrices of the irreducible representations. The base is constructed from the Young diagram by a simple rule.

H. Samelson (Ann Arbor, Mich.).

Schaerf, Henry M. On unique invariant measures. Bull. Amer. Math. Soc. 54, 540–545 (1948).

Let G be an abstract group and let S be a left and right invariant σ-field of subsets of G. The author assumes a form of Weil's condition (M) [A. Weil, L'intégration dans les groupes topologiques et ses applications, Actual. Sci. Ind., no. 869, Hermann, Paris, 1940, p. 141; these Rev. 3, 198] and deduces results concerning the existence of a unique left invariant measure defined on S. The two principal theorems are as follows. (I) If there is a unique left invariant measure m and if $m(A)\leqq m(B)$, then A is (almost) left congruent under a countable partition to a subset of B. (II) A left invariant measure is unique if and only if $m(Xg)$ is absolutely continuous with respect to $m(X)$ for each g in G.

L. H. Loomis (Cambridge, Mass.).

Lyubarskiĭ, G. Ya. On the integration in the mean of almost periodic functions on topological groups. Uspehi Matem. Nauk (N.S.) 3, no. 3(25), 195–201 (1948). (Russian)

It is shown that for any connected locally compact Abelian group G, the mean $M(f)$ of a continuous almost periodic function f on G, as defined by von Neumann, can be expressed as a limit of integrals analogous to those used by Bohr in defining the mean value of an almost periodic function on the reals. Specifically, there is an ascending sequence C_n of compact subsets of G such that for any continuous almost periodic function f on G,

$$\lim_{n\to\infty}(\mu(C_n))^{-1}\int_{C_n} f(g)d\mu(g)$$

exists and equals $M(f)$, where μ denotes Haar measure. This result follows easily from the Pontryagin-van Kampen structure theorem, according to which G is the direct product of a finite-dimensional real vector group and a compact group, a slightly more precise result having previously been obtained in this way by Kawada [Proc. Imp. Acad. Tokyo 19, 264–266 (1943); these Rev. 7, 240] (and as indicated by Kawada, the same result is valid for an arbitrary connected group with a separating family of continuous almost periodic functions, by virtue of the Freudenthal-Weil structure theorem for such groups). The present author, however, makes no use of structure theorems, his proof being based on a lemma proved by a direct argument, and which is also used to give a new proof of the following theorem (due to Gelfand and Raikov [C. R. (Doklady) Acad. Sci. URSS (N.S.) 28, 195–198 (1940); these Rev. 2, 217]). If G is a locally compact Abelian group, then the mapping $g^*\to L$ from the character group G^* of G to the continuous linear functionals on $L_1(G)$, given by $L(f)=\int_G g^*(g)f(g)d\mu(g)$ $(f\in L_1(G))$, is a homeomorphism of G^* into the conjugate space of $L_1(G)$, in its weak topology.

I. E. Segal (Chicago, Ill.).

Rieger, Ladislav S. On ordered and cyclically ordered groups. II. Věstník Královské České Společnosti Nauk. Třída Matemat.-Přírodověd. 1947, no. 1, 33 pp. (1948). (Czech)

Rieger, Ladislav S. On ordered and cyclically ordered groups. III. Věstník Královské České Společnosti Nauk. Třída Matemat.-Přírodověd. 1948, no. 1, 26 pp. (1948). (Czech)

The results of these two parts were abstracted in the English summary which appeared with part I [same Věstník 1946, no. 6 (1947); these Rev. 9, 7].

Cotlar, M., and Zarantonello, E. Semiordered groups and Riesz-Birkhoff L-ideals. Fac. Ci. Mat. Univ. Nac. Litoral. Publ. Inst. Mat. 8, 105–192 (1948). (Spanish)

In addition to some new results, many known facts about partially ordered semigroups are systematized and some-

times generalized. Only commutative vector semigroups are considered. A "regular" semigroup is defined as a semigroup in which (1) $f+g=0$ implies $f=g=0$ and (2) $f\perp g$ and $f\leqq g+h$ imply $f\leqq h$, where $f\perp g$ means that $h\leqq f$ and $h\leqq g$ imply $h=0$. Any partially ordered semigroup which is a lattice is a regular semigroup, and many properties of the former still hold in the latter. For instance, in any regular semigroup S, a Boolean algebra is formed by the set of all lattice-ideals which are "complete" in the sense that any monotone increasing (possibly transfinite) sequence of elements of the ideal which is bounded in S is bounded in the ideal. Various decompositions are also possible in such semigroups.

In § 2, a relation $\leqq^*$ is said to be "subordinate" to the natural order $\leqq$ if (1) $f\leqq g$ implies $f\leqq^* g$; (2) $f_\alpha\leqq^* g$ for all α implies $f\leqq^* g$ (f is the union of the f_α) and dually; (3) $\leqq^*$ is reflexive and transitive; (4) $f\leqq^* g$ if and only if $f+h\leqq^* g+h$. For a given relation $\leqq^*$, there exists Ef (saltus of f) such that $f\leqq^* Ef$, and $f\leqq^* h$ implies $Ef\leqq h$. Then f is called absolutely continuous if $Ef=0$, and singular if $Ef=f$. The sets of functions of these two kinds form complementary ideals in any regular semigroup. For an appropriate choice of $\leqq^*$, this generalizes the Lebesgue theory.

The third section is concerned with representation theory. Freudenthal's spectral theory is developed for vector groups. Also obtained is a representation of Boolean algebras by Borel measurable sets modulo sets of first category. These are combined to yield several theorems about representations of vector groups. For instance, for a positive vector semigroup S to be Archimedean it is necessary and sufficient that S be isomorphic to a subsemigroup (preserving both addition and order) of the semigroup of positive Borel measurable functions, modulo sets of first category, on a totally disconnected compact Hausdorff space.

In § 4, the work of § 2 is extended to nonadditive functions of bounded variation. The upper Kolmogoroff integral Wf of a function f of bounded variation is defined by means of order instead of passing to the limit: Wf is the least additive function with $f\leqq Wf$. *P. M. Whitman.*

Climescu, Al. C. Études sur la théorie des systèmes multiplicatifs uniformes. I. L'indice de non-associativité. Bull. École Polytech. Jassy [Bul. Politehn. Gh. Asachi. Iaşi] 2, 347–371 (1947).

The systems studied in this paper are those finite sets in which the product of two elements is a single element of the set, but no other restrictions are made. There are, of course, n^{n^2} distinct systems of order n. The index of nonassociativity is defined as the number of distinct triples x, y, z, for which $x\cdot yz\neq xy\cdot z$. It is shown by induction that there exist systems with arbitrary index i of nonassociativity, $0\leqq i\leqq n^3$, for $n>2$. Three cases, $n=3$, 4, and 5, are verified in detail in order to start the induction. For $n=2$ the index can take on the values $i=0$, 2, 4, and 8, and the odd values are impossible. *H. Campaigne.*

NUMBER THEORY

***Ore, Oystein. Number Theory and Its History.** McGraw-Hill Book Company, Inc., New York, 1948. x+370 pp. $4.50.

As indicated by the title, this book deals not only with the mathematical aspects of number theory but also with its history. Indeed, the latter aspect occupies a large portion of the book. It is the author's aim to present "the results of the theory integrated more fully in the historical and cultural framework" than is done in the usual texts. In making a selection of topics, those "of systematic and historical importance capable of simple presentation have been preferred." The author has also limited himself to topics requiring only a limited mathematical knowledge.

The portions of the book dealing with the historical and cultural settings of the development of the theory make quite interesting reading. Here one finds reproductions of such things as finger symbols used to denote numbers, number symbols used by various cultures, a small section of the Papyrus Rhind, and two Babylonian tablets, one of which is a table of reciprocals and the other is a list of right triangles with integral sides. Another feature is that a number of quotations from early mathematical writers are given together with problems posed and solved by them. The effect of this treatment is that the reader gains a clear picture of the existing state of knowledge and methods at the various periods. These methods are frequently contrasted with modern methods. The author is also careful to point out still existing lacunae in our knowledge and frequently mentions the results of modern research in the analytic theory of numbers. The references at the ends of the chapters are also very helpful.

With regard to the mathematical material itself, there are a number of topics presented that are not usually found in the texts. One finds, for example, a discussion of the factoring of numbers by the methods of Euler and Fermat as well as tests for primality based on the congruence $a^{n-1}\equiv 1 \pmod n$. There are brief discussions of Fermat numbers (i.e., integers n for which the preceding congruence holds for all a relatively prime to n) and of amicable, abundant, deficient and multiply perfect numbers together with their numerological significance. The author gives a neat proof of Thue that a prime of the form $4n+1$ can be represented as the sum of two squares. He also discusses the function $\lambda(n)$ which is the minimal exponent such that $a^{\lambda(n)}\equiv 1 \pmod n$ for all a relatively prime to n and applies this to a problem in telephone splicing. Also included are indications of the impossibility of trisecting an angle, duplicating the cube and constructing any regular n-gons except those for which n has a certain form; Gauss's construction of the heptadecagon is given.

Nevertheless, one finds much lacking in the way of coverage. For example, no quantitative results on primes are proved, no systematic account of arithmetical functions or any lattice point methods are given, to say nothing of the fact that Gauss's quadratic reciprocity law is not even mentioned. Yet the author has considered it advisable to discuss rings, moduls, lattices, etc. The pace of the book is somewhat slow as a result of frequent numerical examples and historical comments; one also feels that some economy of exposition could be achieved by a more liberal use of symbolism. On page 274, a proof of Euler's theorem is given which is needlessly complicated by the reduction of one reduced residue system ar_i to another such system r_i' by a division process. On page 85, it is stated that Vinogradov could not decide how large an odd number had to be in order that it be possible to represent it as the sum of three primes but the author neglects to mention that the later work of Walfisz made this possible. On the tenth line of

page 282, the reference number (12–17) was inadvertently omitted.

It would appear that the book will be particularly attractive to those interested in number theory more as a pastime than as a serious discipline with its own special techniques and methods. It is also well suited to a first course in the theory of numbers for students at an early stage of development who may not wish to study the subject further. In its own sphere, the book gives a very readable account of the history of (classical) number theory with much serious mathematical thought. In this respect, it is far superior to the usual histories of mathematics. *L. Schoenfeld.*

Wachs, Sylvain. Sur une propriété arithmétique des nombres de Cauchy. Bull. Sci. Math. (2) 71, 219–232 (1947).

The Cauchy numbers are defined by

$$I_m=\frac{1}{m!}\int_0^1 x(x-1)\cdots(x-m+1)dx,$$

so that $x/\log(1+x)=\sum_0^\infty I_m x^m$. R. Liénard [Intermédiaire Recherches Math. 2, 2 (1946)] computed I_m for $m\leqq 21$ and noted that if $m-2$ is an odd prime then I_m is divisible by $m-2$. In the present paper this property is proved to hold generally; the proof makes use of known properties of the Bernoulli numbers and polynomials. *L. Carlitz.*

Majumdar, Kulendra Narayan. On the divisors of $a\alpha^n+b\beta^m$. Bull. Calcutta Math. Soc. 40, 15–16 (1948).

Let a, b, c, d be positive integers, relatively prime in pairs, and write $q=abcd/Q^2$, so that q is square-free. Then no integer of the form $4kq-1$ divides ac^n+bd^m for any pair of nonnegative integers m and n. This is a generalization of a theorem of S. S. Pillai [J. Indian Math. Soc. (N.S.) 6, 120–121 (1942); these Rev. 4, 210]. *I. Niven* (Eugene, Ore.).

Wright, E. M. Equal sums of like powers. Bull. Amer. Math. Soc. 54, 755–757 (1948).

Let $P(k, s)$ denote the smallest j for which there exists a nontrivial solution of the system

$$\sum_{i=1}^{j}a_i^h=\sum_{i=1}^{j}a_i^h=\cdots=\sum_{i=1}^{j}a_i^h, \qquad h=1,\cdots,k,$$

for given k and s. It is known that $P(k, s)\geqq k+1$ and $P(k, 2)\leqq(k^2+4)/2$. The author now proves that

$$P(k, s)\leqq(k^2+k+2)/2$$

and, for k odd, $P(k, s)\leqq(k^2+3)/2$. He makes the conjecture that $P(k, s)=k+1$ (for $k=2$, 3, 5, and for $s=2$, $k=1, 2, \cdots, 9$, this has been proved).

N. G. W. H. Beeger (Amsterdam).

Ehrhart, p. 855

Chang, K. L. On some Diophantine equations $y^2=x^3+k$ with no rational solutions. Quart. J. Math., Oxford Ser. 19, 181–188 (1948).

Using a method of L. J. Mordell [Arch. Math. Naturvid. 49, no. 6, 143–150 (1947); these Rev. 9, 270] the writer proves that $y^2=x^3+k$ has no rational solutions if k satisfies certain conditions. These conditions are similar to those obtained by Mordell for the same Diophantine equation. However, the values of k satisfying the conditions of the present paper are completely distinct from those satisfying Mordell's conditions. *I. Niven* (Eugene, Ore.).

Gloden, A. Solutions minima de la congruence $X^4+1\equiv 0 \pmod{p^a}$, $a=2$, 3, ou 4, pour $p<10^3$. Euclides, Madrid 8, 126 (1948).

Tables are given of the two least solutions of the quartic congruence mentioned in the title. The moduli considered are not quite as described, however. For $a=3$ and 4, $p=17$ and 41 only. *D. H. Lehmer* (Berkeley, Calif.).

Schwarz, Št. On the equation $a_1x_1^k+a_2x_2^k+\cdots+a_kx_k^k+b=0$ in finite fields. Quart. J. Math., Oxford Ser. 19, 160–163 (1948).

It is proved that for $k\mid p-1$, $a_i\neq 0$, the equation of the title is solvable in $GF(p^n)$. The theorem follows from $\sum(a_1x_1^k+\cdots+a_kx_k^k+b)^{p^n-1}\neq 0$, the summation extending over all $x_i\in GF(p^n)$. To prove the last statement, expand by the multinomial theorem and sum over the x_i.

L. Carlitz (Durham, N. C.).

Vandiver, H. S. Cyclotomic power characters and trinomial equations in a finite field. Proc. Nat. Acad. Sci. U. S. A. 34, 196–203 (1948).

Let a, b, n be integers such that $a, b, a+b$ are all nondivisible by n; let l be an odd prime which is not a factor of n and set $\alpha=e^{2\pi i/l}$, $\beta=e^{2\pi i/n}$, $ln=m$; let q be an odd prime and $q^t\equiv 1 \pmod m$. Let g be a primitive root in the finite field of order q^t. Then set

$$\Psi(\chi\beta)=\sum_h(\chi\beta)^{-kh+(a+b)\,\mathrm{ind}_g(g^h+1)},$$

where h ranges over the integers $0, 1, \cdots, q^t-2$ excepting $\frac{1}{2}(q^t-1)$. Furthermore let $\bar{D}$ be the operator $x(d/dx)$. The following congruence is proved in this paper for $r<l$:

$$\left[\bar{D}^{r-1}\left(\frac{\bar{D}\Psi(\chi\beta)}{\Psi(\chi\beta)}\right)\right]_{x=1}\equiv(-1)^r a^r\sum_h\beta^{-ah}A(k)-b^r\sum_h\beta^{bh}A(k)$$
$$+(-1)^{r+1}(a+b)^r\sum_h\beta^{-(a+b)h}A(k) \pmod l,$$

where $k=1, 2, \cdots, q^t-2$ and $A(k)=k^{r-1}\,\mathrm{ind}_g(g^k-1)$. This can be looked upon as a generalization of an earlier result proved by the author [Amer. J. Math. 47, 140–147 (1925)] as well as of a relation, due originally to Kummer, involving certain power characters. It is stated that applications to trinomial equations in a finite field, as well as applications to power characters, will be given elsewhere.

H. W. Brinkmann (Swarthmore, Pa.).

Davenport, H., and Hall, Marshall. On the equation $ax^2+by^2+cz^2=0$. Quart. J. Math., Oxford Ser. 19, 189–192 (1948).

Let a, b, c be integers, not all of the same sign, each pair of which is relatively prime. The paper gives a new proof of the classical result that the equation $ax^2+by^2+cz^2=0$ has an integral solution other than $x=0$, $y=0$, $z=0$ if the congruences, $A^2\equiv -bc \pmod a$, $B^2\equiv -ca \pmod b$, $C^2\equiv -ab \pmod c$, are soluble. It is first shown that the result follows from the fact that a point (not 0) of a lattice of determinant 1 is always within the interior of the star body $|X^2+Y^2-Z^2|\leqq 1$. The proof of this is obtained by use of lattice point theorems for plane convex domains. *D. Derry.*

Davenport, H. Non-homogeneous ternary quadratic forms Acta Math. 80, 65–95 (1948).

Let $\alpha, \beta, \gamma, \delta, c_1, c_2$ be real, with $\Delta=|\alpha\delta-\beta\gamma|\neq 0$. Then Minkowski [Geometrie der Zahlen, Teubner, Leipzig, 1910] showed that there exist integers x, y such that

$|(\alpha x+\beta y+c_1)(\gamma x+\delta y+c_2)| \leqq \Delta/4$. This can be reformulated to assert that for any indefinite binary quadratic form $Q(x, y)$ and any real x_0, y_0, there exist integers x, y such that $|Q(x+x_0, y+y_0)| \leqq \Delta/4$. In the present paper this is generalized to indefinite ternary quadratic forms $Q(x, y, z)$ of determinant $D \neq 0$. It is shown that for any real x_0, y_0, z_0 there are integers x, y, z such that $|Q(x+x_0, y+y_0, z+z_0)| \leqq (27|D|/100)^{\frac{1}{2}}$, and that the constant is best possible. If, however, Q is restricted to null forms, it is shown that 27/100 can be replaced by $\frac{1}{4}$, which again is best possible. It is also shown that except for forms which are equivalent to a multiple of $x^2+5y^2-z^2+5yz+xz$, there are integers x, y, z such that $|Q(x+x_0, y+y_0, z+z_0)| \leqq (1-\delta)(27|D|/100)^{\frac{1}{2}}$ for some $\delta \geqq 10^{-10}$. This result has no analogue for binary quadratic forms [Nederl. Akad. Wetensch., Proc. 49, 815–821 = Indagationes Math. 8, 518–524 (1946); these Rev. 8, 444].

In the course of the proof of the first theorem stated above, the following generalization of Minkowski's theorem is obtained: if μ, ν are positive numbers with $\mu\nu \geqq \frac{1}{16}$, then for any c_1, c_2 there are integers x, y such that

$$-\nu\Delta \leqq (\alpha x+\beta y+c_1)(\gamma x+\delta y+c_2) \leqq \mu\Delta.$$

W. J. LeVeque (Cambridge, Mass.).

Mahler, K. On the admissible lattices of automorphic star bodies. Acad. Sinica Science Record 2, 146–148 (1948).

A star body K of points X for which $F(X) \leqq 1$ is defined to be automorphic if a group Γ of linear substitutions exists, each of which maps K into itself, together with a hypersphere Σ, $|X| \leqq c$, such that each point of K may be mapped into a point of Σ by an appropriate member of Γ. Let d_P denote the set of K-admissible lattices Λ each of which has the property: lower bound of $F(P)=1$, P in Λ, $P \neq 0$. The author proves that the set of determinants of lattices of d_P is closed if K is automorphic. *D. Derry.*

Jarník, Vojtěch. On the main theorem of the Minkowski geometry of numbers. Časopis Pěst. Mat. Fys. 73, 1–8 (1948). (English. Czech summary)

Let K be a convex area in the plane, symmetrical about the origin, bounded by a curve with a continuous radius of curvature which is never less than ρ_1. Let Λ be a plane lattice of determinant 1, and let τ_i, for $i=1$ or 2, be the least number such that K, when magnified by a factor τ_i, contains at least i independent points of Λ. Then van der Corput and Davenport [Nederl. Akad. Wetensch., Proc. 49, 701–707 = Indagationes Math. 8, 409–415 (1946); these Rev. 8, 317] proved a result equivalent to $\tau_1^2 \leqq (1+\delta\rho_1^2)^{-1}$, where $\delta = \frac{1}{2}3^{\frac{1}{2}} - \frac{1}{4}\pi$. The author now proves that

$$\tau_1\tau_2(1+\delta\rho_1^2)(1+\epsilon\rho_1^2) \leqq 1+(1-\tau_1/\tau_2)\delta\rho_1^2+(\tau_1/\tau_2)\epsilon\rho_1^2,$$

where $\epsilon = 1-\frac{1}{4}\pi$; this is more precise if $\tau_1 < \tau_2$.

H. Davenport (London).

Jarník, Vojtěch. On the successive minima of arbitrary sets. Časopis Pěst. Mat. Fys. 73, 9–15 (1948). (English. Czech summary)

Let M be any set of points in n-dimensional space, and M_1 the set of all points $\frac{1}{2}(x-y)$, where x and y are points of M. The author studies various definitions of successive minima, possible generalizations of Minkowski's definition [Geometrie der Zahlen, Leipzig, 1896–1910, chap. 5] for a convex set, symmetrical about the origin. He defines λ_i, μ_i, ν_i, π_i for $i=1, 2, \cdots, n$ as the lower bounds of all positive numbers α such that (i) the union of all sets βM_1 with $\beta \leqq \alpha$, (ii) the set αM_1, (iii) every set βM_1 with $\beta > \alpha$, (iv) the common part of all sets βM_1 with $\beta \geqq \alpha$, respectively, contains at least i linearly independent points with integral coordinates. Here αM_1 denotes the set of all points αx, where x is in M_1. It follows from these definitions that each of the four sets of numbers is monotonic increasing (in the wide sense), and that $\lambda_i \leqq \mu_i \leqq \nu_i \leqq \pi_i$. Various inequalities are known which generalize partially Minkowski's inequality for the product of the successive minima [Jarník, Věstník Královské České Společnosti Nauk. Třida Matemat.-Přírodověd., 1941; Jarník and Knichal, Rozpravy II. Třídy České Akad. 53, no. 43 (1943); these Rev. 7, 417; 8, 565; C. A. Rogers, forthcoming paper]. The aim of the present paper is to show that some of these cannot be essentially improved. For example, it is known that

$$\mu_1 \cdots \mu_{i-1}\nu_i\mu_{i+1} \cdots \mu_n L(M) \leqq 2^{2n-1},$$

where $L(M)$ denotes the inner Lebesgue measure of M; and the author constructs, for any T, i, j a set M for which

$$\lambda_1 \cdots \lambda_{i-1}\nu_i\lambda_{i+1} \cdots \lambda_{j-1}\nu_j\lambda_{j+1} \cdots \lambda_n L(M) > T.$$

Other results relate to the set M_p, obtained by "differencing" p times. *H. Davenport* (London).

Negoescu, Nicolae. Théorèmes sur des approximations asymétriques. C. R. Acad. Sci. Paris 226, 1664–1666 (1948).

Continuation d'une note antérieure [mêmes C. R. 226, 1495–1497 (1948); ces Rev. 9, 569]. L'auteur énonce quelques théorèmes qui sont des généralisations des théorèmes de Perron et d'autres [voir Koksma, Diophantische Approximationen, Ergebnisse der Math., v. 4, no. 4, Springer, Berlin, 1936, chap. III, § 2] et de B. Segre [Duke Math. J. 12, 337–365 (1945); ces Rev. 6, 258] et qui sont liés à quelques théorèmes de R. M. Robinson [Bull. Amer. Math. Soc. 53, 351–361 (1947); ces Rev. 8, 566].

J. F. Koksma (Amsterdam).

Hinčin, A. Ya. Dirichlet's principle in the theory of Diophantine approximations. Uspehi Matem. Nauk (N.S.) 3, no. 3(25), 3–28 (1948). (Russian)

An expository survey of the Dirichlet "Schubfachprinzip" and its applications in linear Diophantine analysis. In chapter 1 the principle is applied to prove various known theorems concerning solutions of sets of homogeneous linear inequalities; by using a generalised version of the standard Dirichlet principle, the treatment of limiting cases is somewhat facilitated. In chapter 2 the principle is applied to nonhomogeneous linear inequalities; by a method due originally to Mordell, theorems are proved correlating the solubility of the nonhomogeneous with that of the corresponding homogeneous inequalities. *F. J. Dyson.*

Eichler, Martin. Grundzüge einer Zahlentheorie der quadratischen Formen im rationalen Zahlkörper. I. Comment. Math. Helv. 20, 9–60 (1947).

Eichler, Martin. Grundzüge einer Zahlentheorie der quadratischen Formen im rationalen Zahlkörper. II. Comment. Math. Helv. 21, 1–28 (1948).

Brandt has pointed out, with many illustrations, the relative simplicity of the arithmetical theory of quadratic stem-forms, as compared with the usual theory which treats stem-forms and others alike, or in some cases excludes stem-forms by fixing attention on so-called classical forms. [I: Jber. Deutsch. Math. Verein. 47, 149–159 (1937); II: Festschrift zum 60. Geburtstag von Prof. Dr. Andreas

Speiser, 87–104, Füssli, Zurich, 1945; these Rev. 8, 138.] In his studies of the arithmetic of quaternions [cf. III: Jber. Deutsch. Math. Verein. 53, 23–57 (1943); these Rev. 8, 198], Brandt has also considered simultaneously all ideals belonging to a quaternion order (integral set) and the corresponding system of quaternary quadratic forms. These are stem-forms when the order is maximal. Eichler develops here an arithmetical theory of quadratic forms, in any number of variables, in which stem-forms only are dealt with for the most part, and in which an appropriate system of forms is simultaneously studied. He has already initiated the theory, chiefly for an even number of variables, in two earlier papers [IV: S.-B. Math.-Nat. Abt. Bayer Akad. Wiss. 1943, 1–24; V: same Festschrift as in II, 34–36; these Rev. 9, 136; 7, 369]. In particular he has introduced [V] the concept of a transformer, which is here expanded into an "ideal" theory, and by means of which a system of forms is effectively studied.

A fundamental basis for the theory is the work of Hasse on quadratic forms over a number field [cf. VI: J. Reine Angew. Math. 153, 158–162 (1924)] and that of Witt on the same subject, which leads to a group of form types like Brauer's group of algebra classes [VII: J. Reine Angew. Math. 176, 31–44 (1937)]. The recent analytical studies of quadratic forms by Hecke [cf. VIII: Danske Vid. Selsk. Math.-Fys. Medd. 17, no. 12 (1940); these Rev. 2, 251] and Siegel [cf. IX: Ann. of Math. (2) 36, 527–606 (1935)] also indicate a certain parallelism between the theory of forms and algebras, from a different point of view, and provide one of Eichler's principal objectives for the application of his arithmetical theory. This is Hecke's theorem on matrices of representation numbers by members of a system of definite forms, called Hecke matrices by Brandt [cf. III for 2 and 4 variables]. A second principal objective is the generalization of Meyer's theorem on the equality of class-number and genus-number in the indefinite case.

The first chapter deals with the theory of types and genera. A quadratic form in n variables $f^{(n)}=f(x_1, \cdots, x_n)$ is written as a matrix product $f^{(n)}=\mathfrak{F}^{(n)}=x\mathfrak{F}x$, $\mathfrak{F}=(f_{ij})$, $f_{ij}=f_{ji}$, and f_{ii} even if the form is integral. The discriminant [cf. I] is defined to be

$$D(\mathfrak{F})=D=(-1)^m|\mathfrak{F}^{(2m)}|=(-1)^m|f_{ij}|, \qquad n=2m,$$
$$D(\mathfrak{F})=D=\tfrac{1}{2}(-1)^m|\mathfrak{F}^{(2m+1)}|=\tfrac{1}{2}(-1)^m|f_{ij}|, \qquad n=2m+1.$$

The form $\mathfrak{F}_0^{(2m)}$, with the matrix consisting of 1's on the secondary diagonal and zeros elsewhere, is in the neutral element of Witt's group. Two forms $\mathfrak{F}_1^{(n_1)}$ and $\mathfrak{F}_2^{(n_2)}$ are of the same type if $n_1 \equiv n_2 \pmod 2$ and

$$\mathfrak{F}(x_1, \cdots, x_{n_1})-\mathfrak{F}(x_{n_1+1}, \cdots, x_{n_1+n_2})$$

is in the same genus as $\mathfrak{F}_0^{(n_1+n_2)}$. Two forms F_1 and F_2 in the same number of variables are said to be in the same genus if there exists a nonsingular matrix S such that $F_1=\hat{S}F_2S$, and in the same class if such a relation holds with an integral S whose determinant is a unit. The author determines two sets of genus invariants, the second of which involves characters but is independent of order invariants ("order" in the usual form theory sense). The theory of genera is not restricted to stem-forms, but the chapter is concluded with a characterization of them in terms of discriminants and characters. The cases of an even and an odd number of variables exhibit nontrivial differences.

The second chapter deals with the ideal theory of form systems, where a system is the totality of stem-forms having the same number of variables, signature and discriminant. Representatives of the classes in a system are considered and transformers from one to another of these are studied. Some numbers of ideals of various prime power norms are computed for use in Chapters 3 and 4. Vectors in Witt's metric space [VII] associated with a genus of forms are also studied and various numbers of them are computed.

In the third chapter [part II], Eichler partly attains his first principal objective. By his new methods, he obtains a proof of some of Hecke's results [VIII] for an even number of variables, but he states that he has not been able so far to find the connection between his results and Hecke's in the case of primes for which $(D/p)=-1$. He also obtains a result of Siegel [IX] on representation weights.

In the fourth chapter Eichler attacks his second principal objective. He proves a theorem on the existence of principal transformers in an extension field, which is a generalization of a theorem of Wedderburn. He studies units of indefinite forms. He does not completely achieve his objective, but states that he has in mind a method to complete an induction proof if Meyer's method [Vierteljschr. Naturforsch. Ges. Zürich 36, 241–250 (1891)], inaccessible to him at the time of writing, will not go through. *R. Hull.*

*Eichler, M. **Zahlentheorie der quadratischen Formen.** Ber. Math.-Tagung Tübingen 1946, pp. 63–64 (1947).
Cf. the preceding review.

Ko, Chao, and Chu, Fu-Tsu. On the equivalence of positive definite quadratic and Hermitian forms. Acad. Sinica Science Record 2, 148–155 (1948).

Let $f_n(x)=\sum_{i,j=1}^{n}a_{ij}\bar{x}_ix_j$, $a_{ij}=\bar{a}_{ji}$, be a quadratic or Hermitian form according as the coefficients and variables are real or complex. Let $x_i=\sum\alpha_{ij}y_j$ be a unimodular transformation with α_{ij} rational integers, or quadratic integers of the field $k(\theta)\subset F$. Two positive definite forms $f_n(x)=a\bar{x}_1x_1+f_{n-1}(x)$, $g_n(x)=a\bar{y}_1y_1+g_{n-1}(y)$ are equivalent if and only if $f_{n-1}(x)\sim g_{n-1}(y)$. If $A(x_1, \cdots, x_s)$, $B(x_{s+1}, \cdots, x_n)$, $C(y_1, \cdots, y_s)$, $D(y_{s+1}, \cdots, y_n)$ are positive definite forms with integral coefficients, either rational or in $k(\theta)$, and if $A\sim C$ and A is indecomposable, then $A+B\sim C+D$ if and only if $B\sim D$. *C. C. MacDuffee* (Madison, Wis.).

Ko, Chao. On the decomposition of quadratic forms in eight variables. Sci. Rep. Nat. Tsing Hua Univ. 4, 337–340 (1947).

This article apparently preceded one already reviewed [Acad. Sinica Science Record 1, 33–36 (1942); these Rev. 5, 201] but was delayed in publication. It modifies slightly the proof of the theorem given in the cited article.
G. Pall (Chicago, Ill.).

Latimer, Claiborne G. Quaternion algebras. Duke Math. J. 15, 357–366 (1948).

With a rational generalized quaternion algebra are associated various sets of integers α, β, γ such that $\alpha\beta\gamma$ is square-free, and such that a basis of the algebra is $1, i, j, k$, where $i^2=-\beta\gamma$, $j^2=-\gamma\alpha$, etc. Equivalent algebras are obtained by applying rational linear transformations to i, j, k. Two algebras are shown to be equivalent if and only if they have the same fundamental number $d=2^e fD$, where $e=0$ or 1 according as $(\alpha+\beta)(\beta+\gamma)(\gamma+\alpha)(\alpha+\beta+\gamma)/8$ is even or odd, $f=+1$ or -1 according as the form $\phi=\alpha x^2+\beta y^2+\gamma z^2$ is definite or indefinite, and $D=ABC$, where A, B, C are the least positive integers such that each of the congruences $x^2\equiv-\beta\gamma \pmod{\alpha/A}$, $y^2\equiv-\gamma\alpha \pmod{\beta/B}$, $z^2\equiv-\alpha\beta \pmod{\gamma/C}$ is solvable. [Using the Hasse invari-

ants c_p of the quadratic form ϕ, the reviewer notes that $d=\pm\prod p$, where p ranges over the primes ($p\geqq 2$) such that $c_p(\phi)=-1$.] As an application, the conditions for a ternary or quaternary quadratic form to represent zero rationally are considered. *G. Pall* (Chicago, Ill.).

Skolem, Th. On the existence of a multiplicative basis for an arbitrary algebraic field. Norske Vid. Selsk. Forh., Trondhjem **20**, no. 2, 4–7 (1947).

It is proved that the multiplicative group of the nonzero numbers in an algebraic field has a set of independent generators even if the class number h of the field exceeds 1. Such a set is in fact constructed. Examination of the representations of the integers of the field in terms of these generators displays the difference between the cases $h=1$ and $h>1$. *O. Todd-Taussky* (London).

Behrend, F. A. Generalization of an inequality of Heilbronn and Rohrbach. Bull. Amer. Math. Soc. **54**, 681–684 (1948).

Let

$$T(a_1,\cdots,a_m)=1-\sum_{1\leqq\mu\leqq m}1/a_\mu+\sum_{1\leqq\mu<\nu\leqq m}1/\{a_\mu,a_\nu\}-\cdots+(-1)^m 1/\{a_1,\cdots,a_m\},$$

where $\{u_1,\cdots,u_m\}$ denotes the least common multiple of the positive integers $u_1,\cdots,u_m$. Rohrbach [J. Reine Angew. Math. **177**, 193–196 (1937)] and the reviewer [Proc. Cambridge Philos. Soc. **33**, 207–209 (1937)] proved: $T(a_1,\cdots,a_m)\geqq(1-1/a_1)\cdots(1-1/a_m)$. The author proves the stronger inequality $T(a_1,\cdots,a_m,b_1,\cdots,b_n)\geqq T(a_1,\cdots,a_m)T(b_1,\cdots,b_n)$. His proof is based on the following lemma. If $(d,v_1)=\cdots=(d,v_l)=1$, then $T(du_1,\cdots,du_k,v_1,\cdots,v_l)=(1/d)T(u_1,\cdots,u_k,v_1,\cdots,v_l)+(1-1/d)T(v_1,\cdots,v_l)$.

Assuming that a_μ does not divide a_ν $(\mu\neq\nu)$, b_μ does not divide b_ν $(\mu\neq\nu)$, the author shows that

$$T(a_1,\cdots,a_m,b_1,\cdots,b_n)=T(a_1,\cdots,a_m)T(b_1,\cdots,b_n)$$

if and only if the products $a_1\cdots a_m$ and $b_1\cdots b_n$ are relatively prime. *H. Heilbronn* (Bristol).

Gupta, Hansraj. A table of values of $\tau(n)$. Proc. Nat. Inst. Sci. India **13**, 201–206 (1947).

A table of the first 400 coefficients $\tau(n)$ in the power series $\{(1-x)(1-x^2)\cdots\}^{24}=\sum_{n=1}^{\infty}\tau(n)x^{n-1}$ is given. [For earlier tables of this function see Lehmer, Duke Math. J. **10**, 483–492 (1943); Gupta, Proc. Benares Math. Soc. **5**, 17–22 (1943); these Rev. **5**, 35; **7**, 50.] Six congruence properties due to various authors are quoted. *D. H. Lehmer.*

Wintner, Aurel. On an oscillatory property of the Riemann Ξ-function. Math. Notae **7**, 177–178 (1947).

The author deals with properties of

$$\Xi(t)=\tfrac{1}{2}s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s),$$

where $s=\frac{1}{2}+it$ and $\zeta(s)$ is the Riemann ζ-function. He proves that $\int_0^t\Xi(u)du$ and the "conjugate" of $\Xi(t)$ are both positive for $0<t<\infty$, deducing this from the known fact [Hurwitz-Wintner] that $\phi(x)$, the Fourier cosine transform of $(2/\pi)^{\frac{1}{2}}\Xi(t)$, is positive and decreasing for $0\leqq x<\infty$. *H. Kober* (Birmingham).

Brun, Viggo, Jacobsthal, Ernst, Selberg, Atle, and Siegel, Carl. Correspondence about a polynomial which is related to Riemann's zeta function. Norsk Mat. Tidsskr. **28**, 65–71 (1946). (Norwegian and German)

The Euler-Maclaurin sum formula applied to the series $\sum n^{-s}$ gives (formally) the divergent expansion

$$(s-1)\zeta(s)=\sum_{n=0}^{\infty}B_n(s-1)s(s+1)\cdots(s+n-2)/n!=\sum_{n=0}^{\infty}P_n(s),$$

where $B_0, B_1, B_2, \cdots$ are the Bernoulli numbers, except B_1 which is $+\frac{1}{2}$. The first author observes that, if $Q_n(s)=\sum_0^n P_k(s)$, we may write

$$(2m)!Q_{2m}(s)=B_{2m}(s+2)(s+4)\cdots(s+2m)(s+2m+1)q_{2m}(s),$$

where $q_{2m}(s)=s^{m-1}+\cdots$ is a polynomial, so that $Q_{2m}(s)$ has m real zeros in common with $\zeta(s)$; and he then asks whether the $m-1$ zeros of $q_{2m}(s)$ have anything to do with zeros of $\zeta(s)$. The second and third authors note that the observation about real zeros is a simple deduction from the formula for the remainder $R_n(s)=(s-1)\zeta(s)-Q_n(s)$ given by Stieltjes, and are sceptical about any deeper connection involving complex zeros. The fourth author substantiates this view by showing, with the aid of Rouché's theorem, that the $2m$ zeros of $Q_{2m}(s)$ all lie inside a set of circles of radius 11 about the points $s=-1,-2,\cdots,2-2m$; and states that, by a more accurate estimation of P_k/P_{2m}, it can be shown that, when m is large, the zeros of $Q_{2m}(s)$ are real and approximate to those of $P_{2m}(s)$, with the possible exception of a bounded number, which in any case lie inside a circle of fixed radius about $s=-2m$. *A. E. Ingham.*

Kössler, M. Asymptotic expansions for the functions $\zeta(s)$ and $\zeta(a,s)$. Rozpravy II. Třídy České Akad. **51**, no. 32, 10 pp. (1941). (Czech)

The author gives asymptotic expansions for

$$\zeta(2v+1)=\sum_{n=1}^{\infty}n^{-2v-1},\quad S(2v)=\sum_{n=0}^{\infty}(-1)^n(2n+1)^{-2v}$$

(v a positive integer). They are based on asymptotic expansions for $\zeta(s)$ and $\zeta(a,s)$. For example, let $N>0$, $K\geqq 0$ be integers, $\Re(s)<2K+3$. Then

$$2(2\pi)^{-s}(1-2^s)\Gamma(s)\cos\tfrac{1}{2}\pi s\,\zeta(s)=\sum_{k=1}^{N}(-1)^{k+1}k^{s-1}+\sum_{k=0}^{K}(-1)^{k+N}E_k 2^{-2k-1}\binom{s-1}{2k}(N+\tfrac{1}{2})^{s-1-2k}+R(N,K),$$

where

$$2^{2K+3}(2K+2)!\Gamma(1-s)R(N,K)=(-1)^{N+K+1}E_{K+1}\int_0^{\infty}e^{-(N+\frac{1}{2})z}z^{2K+2-s}\theta(z)dz$$

($0<\theta(z)\leqq 1$; E_i, Euler's numbers). For $s=2v+1$ the left side is zero and it is necessary to calculate the derivative in order to obtain $\zeta(2v+1)$. Some of the asymptotic formulae also lead to exact formulae (for $N\to\infty$) which are analogous to Wallis's and Stirling's formulae, e.g.,

$$\frac{1}{4\pi^2}\zeta(3)=\lim_{n\to\infty}\log\frac{(1,2)(2,2)\cdots(n-1,2)}{\exp\{\frac{1}{4}n(n-1)(2n-1)\log n+\frac{1}{12}n\}},$$

where $(k,s)=k^{k^s}$, $\exp x=e^x$. *V. Jarník* (Prague).

Guinand, A. P. A summation formula in the theory of prime numbers. Proc. London Math. Soc. (2) **50**, 107–119 (1948).

The Riemann hypothesis is assumed to be true. Let $N(t)$ denote the number of zeros $s=\frac{1}{2}+i\gamma$ of the Riemann zeta

function $\zeta(s)$, satisfying $0<\gamma<1$; the Hankel transform $2\pi G(x)$ of order $\frac{3}{2}$ of the function

$$2\pi F(x)=1-\log(x/2\pi)+2\pi N(x)/x$$

is found explicitly in terms of $\sum p^{-\frac{1}{2}m}\log p$ (summed over $0<m\log p\leqq x$) and elementary functions. For suitable functions $f(x)$, the functions $xf'(x)$ and $xg'(x)$ are a pair of Hankel transforms of order $\frac{3}{2}$, when $g(x)$ is the cosine transform of $f(x)$ [Guinand, Ann. of Math. (2) 42, 591–603 (1941); these Rev. 3, 109]. These two results, when combined with the Parseval relation $\int_0^\infty xf'(x)G(x)dx=\int_0^\infty xg'(x)F(x)dx$, lead to summation formulae involving $\sum p^{-\frac{1}{2}m}(\log p)f(m\log p)$ summed over $0<m\log p<x$ and $\sum g(\gamma)$ summed over $0<\gamma<x$.

P. Hartman (Baltimore, Md.).

Mirsky, L. On a theorem in the additive theory of numbers due to Evelyn and Linfoot. Proc. Cambridge Philos. Soc. 44, 305–312 (1948).

The error term $O(n^{s-2+s/[(s-1)r+1]+\epsilon})$ is obtained in place of the $O(n^{s-2+1/r+(s-1)/r^2+\epsilon})$ in the result of Evelyn and Linfoot for the number of representations of n as the sum of s r-free integers which are congruent to k(mod q) [Quart. J. Math., Oxford Ser. 4, 309–314 (1933)]. This is an improvement for $s\geqq 3$. *W. H. Simons* (Vancouver, B. C.).

Erdös, P. On the density of some sequences of integers. Bull. Amer. Math. Soc. 54, 685–692 (1948).

Let $a_1<a_2<\cdots$ be a sequence of integers such that no one divides any other, and let $b_1<b_2<\cdots$ be the sequence consisting of all integers divisible by at least one of the a's. Then a necessary and sufficient condition that the b's have a density is that

$$(1)\quad \lim_{\epsilon\to 0}\limsup_{n\to\infty} n^{-1}\sum_{n^{1-\epsilon}<a_i\leqq n}\varphi(n;a_i;a_1,\cdots,a_{i-1})=0,$$

where $\varphi(n;x;y_1,\cdots,y_n)$ denotes the number of integers not exceeding n which are divisible by x but not by $y_1,\cdots,y_n$. In particular, if $f(n)$, the number of a's not exceeding n, satisfies the condition $f(n)<cn/\log n$, where c is a constant, then the b's have a density. The result (1) is also used to show that the density of the set of integers m having two divisors d_1, d_2 satisfying $d_1<d_2<2d_1$ exists. The paper concludes with some unsolved problems.

W. H. Simons (Vancouver, B. C.).

ANALYSIS

Calculus

Murnaghan, Francis D. Introduction to Applied Mathematics. John Wiley & Sons, Inc., New York; Chapman & Hall, Limited, London, 1948. ix+389 pp. $5.00.

This text for graduate students treats a well selected group of topics. These include vectors and matrices in 2, 3 and n-space; the axis problem; Fourier series; boundary-value problems leading to Legendre, Bessel and other functions; integral equations; the calculus of variations and dynamics; and the operational calculus. For a book on this scale there is a good balance between precision and motivation. In a few cases brief proofs are achieved by relegating major portions to the exercises, as in the proof for convergence in the mean and convergence of Fourier series. Omitted minor points are often covered by the use of (why?), (show this), (check this) and similar asides. The notation is brief and consistently used, but at times may strain the beginner, as when he reads $p\delta x$ and must recall that p means $p_j=\partial F/\partial\dot{x}_j$, δx means δx_j, and a doubled j means summation so that $p\delta x$ means $\sum\partial F/\partial\dot{x}_j\delta x_j$. Numerous exercises are provided, with no lack of those requiring original thinking, and many hints or solutions for the more difficult questions are given. For the student willing to master the notation, and accept the condescension implied by the informal warnings and admonitions, this should make an excellent basis for self study or for a course.

P. Franklin (Cambridge, Mass.).

Smith, T. A series for the stationary value of a function. Proc. Phys. Soc. 59, 323–326 (1947).

Let $f(x_i)$ be a function of n arguments x_1 to x_n. Let $F=f(X_i)$ be a value where all the first partial derivatives f_i are zero. Then the series in question enables one to calculate the value F from the value of f and all its first and higher partial derivatives at a point near X_i, where the f_i are small. Neither the values of x_i nor of X_i are involved.

P. Franklin (Cambridge, Mass.).

Ghosh, P. K. On (C,α)-convergent integrals and their application to mathematical physics. Bull. Calcutta Math. Soc. 40, 1–7 (1948).

The author extends the discussion of his earlier paper [same Bull. 39, 19–29 (1947); these Rev. 9, 274] from $(C,1)$ to (C,α) summability. *R. P. Boas, Jr.*

Theory of Sets, Theory of Functions of Real Variables

Hadwiger, H. Multikongruenz ebener Mengen und Pythagoreischer Lehrsatz. Bull. École Polytech. Jassy [Bul. Politehn. Gh. Asachi. Iaşi] 2, 98–105 (1947).

Though the principal definition of this paper is stated in a generalized form (relating to metric spaces), the results apply only to "polygons" in the Euclidean plane. By "polygon" is understood the sum of the interior of an ordinary polygon and any chosen subset of its boundary. Two polygons are congruent by decomposition relative to the family K of polygons (briefly, multicongruent with respect to K) if they are decomposable into a finite number of disjoint polygons (i.e., without common points) which are congruent in pairs (in the sense of superposability). It is the disjointness of the components in the set-theoretic sense, i.e., even with respect to the boundary points belonging to them (a condition not customarily required) which leads to new problems and, in particular, to those of the present note. It is proved that for two polygons A and B to be congruent by decomposition with respect to K, it is necessary that they have equal areas, and that $L(A)-2L^*(A)=L(B)-2L^*(B)$, where $L(A)$ is the length of the boundary of A, and $L^*(A)$ a Banach linear measure for the boundary points of A which belong to it. A half-closed rectangle $A:(0\leqq x<a,\ 0\leqq y<b)$ and a half-closed square $B:(0\leqq x<c,\ 0\leqq y<c)$ are congruent by decomposition with respect to K if $c^2=ab$; the contrary holds if A (with $a\neq b$) is closed (or

open). The proof is made by means of a rotation and $\aleph_0$ translations of A (or B) with resulting rectangles (squares) covering every point of the plane just once. There are refined extensions of the Pythagorean theorem.

H. Blumberg (Columbus, Ohio).

Robinson, Raphael M. On the decomposition of spheres. Fund. Math. 34, 246–260 (1947).

It is shown that the surface S of a sphere may be decomposed into four disjoint sets each congruent to S (in the sense of superposition, via translations and rotations). An immediate consequence is that a solid sphere may be similarly decomposed into five pieces, the number 5 not being reducible. This minimal property supersedes earlier results of Banach and Tarski, von Neumann, and Sierpiński; nor can the number 5 be reduced on allowing reflections. The argument is effected by means of the proved existence of m "independent" rotations in 3-space (m any integer), and certain properties of the group which these rotations generate. The results of others are not utilized except one due to Hausdorff [Grundzüge der Mengenlehre, Leipzig, 1914, pp. 469–472].

H. Blumberg (Columbus, Ohio).

Sierpiński, W. Le paradoxe de Hausdorff et le paradoxe de Banach et Tarski. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 4, 270–272 (1948).

Two sets are congruent by finite decomposition if they are decomposable into a finite number of disjoint subsets which are congruent in pairs (in the sense of superposability). Hausdorff proved [Grundzüge der Mengenlehre, Leipzig, 1914] that if a denumerable set is neglected, the surface of a sphere is congruent, by finite decomposition, to that of two spheres of the same radius. A similar result follows readily for the solid sphere. Banach and Tarski [Fund. Math. 6, 244–277 (1924)] gave a complicated proof of the latter result, the allowance of the neglect of points being eliminated. The present paper gives a brief and elementary proof of this result of Banach and Tarski, using the following lemma. If S is a sphere, and D a finite or denumerable subset of S, then S and $S-D$ are congruent by finite decomposition, the number of components being 2. [Cf. the preceding review.]

H. Blumberg.

Sierpiński, W. Sur une propriété de la droite qui résulte de l'hypothèse du continu. Mathematica, Timişoara 23, 52–53 (1948).

Proof by means of the continuum hypothesis, of the theorem that a straight line L is the sum of $2^{\aleph_0}$ disjoint sets E, each congruent to L by denumerable decomposition; i.e., L and each E are decomposable into $\aleph_0$ disjoint components: $L=\sum_{n=1}^{\infty}L_n$, $E=\sum_{n=1}^{\infty}E_n$, such that, for every n, L_n is congruent to E_n (in the sense of superposability). The argument uses a result of Banach and Tarski [Fund. Math. 6, 244–277 (1924)]. A corollary is the validity, under the continuum hypothesis, of a like result when L is an n-dimensional Euclidean space, $n>1$. For $n\geqq 3$, this result holds (as proved in an earlier paper of the author by a lengthier argument [Fund. Math. 33, 235–244 (1945); these Rev. 8, 140]) without recourse to the continuum hypothesis, congruence being taken in the sense of finite decomposition.

H. Blumberg (Columbus, Ohio).

Borel, Émile. Sur l'addition vectorielle des ensembles de mesure nulle. C. R. Acad. Sci. Paris 227, 103–105 (1948).

Borel, Émile. Sur l'addition vectorielle d'une infinité dénombrable d'ensembles de mesure nulle. C. R. Acad. Sci. Paris 227, 167–168 (1948).

The vector sum of sets E_n of real numbers is the set of numbers $\sum x_n$, where $x_n \epsilon E_n$. For certain sets E of Cantor type (ensembles à définition numérique) the logarithmic rarefaction $\rho(E)$ is defined: for such sets it equals the Hausdorff dimenson $d(E)$. Then for the vector sum of a finite number of such sets E_n to be of zero measure it is sufficient that $\sum\rho(E_n)<1$. The definition of $\rho(E)$ and the theorem are extended to a wider class of sets. In this class, $\rho(E)$ in general exceeds $d(E)$; and a concept of monotonic functions moderately discontinuous is used to delimit the class.

The second note first refers to extensions to certain sets not nowhere-dense and then indicates by examples that the theorem fails for the vector sum of a sequence of sets.

H. D. Ursell (Leeds).

Borel, Émile. Sur la raréfaction R et l'addition vectorielle des ensembles de mesure nulle. C. R. Acad. Sci. Paris 227, 453–455 (1948).

Borel, Émile. Sur la somme vectorielle des ensembles non parfaits de mesure nulle. C. R. Acad. Sci. Paris 227, 545 (1948).

Let a set E be formed by removing from the closed interval $(0, 1)$ a sequence $\{I_n\}$ of open intervals in decreasing order of their lengths a_n. Let b_n be the length of the shortest of the $n+1$ residual intervals after $I_1, \cdots, I_n$ have been removed. Then $R=\min[b_n/(b_n+a_n)]$ is called the raréfaction relative minimum of E. It is shown that, if two such sets E, E' have rarefactions R, R' and $R+R'\geqq 1$, then the vector sum of E, E' is an interval; if $R+R'<1$ then the vector sum has rarefaction greater than or equal to $R+R'$.

The second note extends the definition and result to any set E^* expressible as the vector sum of a perfect set E and a denumerable set D, and defines a class of Suslin sets G any two of which have an open interval for vector sum. [Cf. the preceding review.]

H. D. Ursell (Leeds).

Ważewski, Tadeusz. Sur l'évaluation du domaine d'existence des fonctions implicites réelles ou complexes. Ann. Soc. Polon. Math. 20 (1947), 81–120 (1948).

Soient $w^i=g^i(u_1, \cdots, u_p, v_1, \cdots, v_p)=g^i(u|v)$, $i=1, \cdots, n$, des fonctions (réelles ou complexes) continues avec les dérivées partielles $g^i_{u_\alpha}=\partial g^i/\partial u_\alpha$, $g^i_{v_j}=\partial g^i/\partial v_j$ ($\alpha=1, \cdots, p$; $j=1, \cdots, q$) dans un ensemble ouvert A des variables (réelles ou complexes). La forme hermitienne

$$\sum_{i=1}^{n}|dw|^2=\sum_{i=1}^{n}\left|\sum_{\alpha=1}^{p}g^i_{u_\alpha}du_\alpha\right|^2,$$

les v_j considerées comme paramètres, est dite la forme métrique des fonctions g^i relative aux variables u. Les racines $s_1(u|v)\leqq\cdots\leqq s_p(u|v)$ de l'équation caractéristique $\det(a_{\alpha,\beta}-\delta_{\alpha,\beta}\cdot s)=0$, où $a_{\alpha,\beta}=\sum_{i=1}^{n}g^i_{u_\alpha}\overline{g^i_{u_\beta}}$, $\delta_{\alpha,\alpha}=1$, $\delta_{\alpha,\beta}=0$ pour $\alpha\neq\beta$, sont non négatives et les nombres $\underline{\text{all}}_\alpha(g; u|v)=\sqrt{s_1(u|v)}$, $\overline{\text{all}}_\alpha(g; u|v)=\sqrt{s_p(u|v)}$ sont l'allongement inférieur et supérieur, au point $(u|v)$, des fonctions g^i, rélatifs aux $u_1, \cdots, u_p$. On définit pareillement $\underline{\text{all}}_\nu(g; u|v)$, $\overline{\text{all}}_\nu(g; u|v)$. Si $q=n$ et si les g^i sont données dans l'ensemble L défini par les inégalités $\sum_{\alpha=1}^{p}|u|^2<r^2\leqq+\infty$,

$\sum_{j=1}^{n}|v|^2<R^2\leqq+\infty$ et l'on a $g^i(0,\cdots,0)=0$ et, dans L, $\underline{all}_u\ (g;\ u|v)\leqq\Omega$, $\underline{all}_v\ (g;\ u|v)\geqq\omega$, $\omega(>0)$ et Ω nombres réelles fixes et finis, dans la sphère $\sum_{i=1}^{n}|u|^2\leqq\rho^2$, où $\rho=\min(r, \omega R/\Omega)$, on peut définir un système (unique) des fonctions $v_j=\psi_j(u_1,\cdots,u_n)$ qui remplissent les $\psi_j(0,\cdots,0)=0$ et les $g^i(u|\psi)\equiv 0$. Ce théorème fournit une évaluation du domaine d'existence des fonctions implicites. L'auteur evalue aussi le domaine d'existence de la transformation inverse d'une transformation donnée et le domaine dans lequel une transformation uniforme est invertible.

G. Scorza Dragoni (Padova).

Cotlar, Mischa, and Frenkel, Yanny. A general theory of integral based on an extension of the concept of limit. Univ. Nac. Tucumán. Revista A. 6, 113–159 (1947). (Spanish)

For a real function $f(s)$ defined on an abstract set S and possibly multivalued, the authors define a general concept of limit, termed P limit, where P is a class of subsets of S, termed sets (p). The notation of this review will depart slightly from that of the paper: a set (p') is the complement of a set (p), and a set $(-p)$ or $(-p')$ is a set not (p) or not (p'); the classes of such sets are P', and $-P$ or $-P'$, respectively. The definition of P limit, and of upper or lower P limit, is derived from that of point of P accumulation: this is a real number, every neighbourhood of which contains a subset of the form $\phi(C)$, where C is a set (p) and $\phi(s)$ is for each s a determination of $f(s)$. There is a duality between the classes P and $Z=-P'$; in the theory of P limits, the oft recurring expressions "for some large integer n" and "for all large integers n" of ordinary analysis, are replaced by "in a set (p)" and "in a set (s)." The class P is termed a basis if P' is hereditary and nonvacuous, and termed compact if $-P$ is finitely additive; in that case Z is a subclass of P.

In the application to the theory of the integral, S is the set $S(I)$ of the subdivisions $s=s(I)$ of an interval I of the straight line, and $f(s)$ is the extension (by addition of the contributions of the meshes) of a function of intervals. Given for each interval I a compact basis $P=P(I)$ on the set $S(I)$, the P limit of $f(s)$ is defined to be the P integral of $f(I)$. Some postulates are needed concerning the dependence of $P(I)$ on I; in order to state them concisely, let $s(e)$ denote a subdivision of a finite set of nonoverlapping intervals e, and let $P(e)$, $Z(e)$ denote the extensions of $P(I)$, $Z(I)$ obtained in the natural way (formation of topological product). The authors introduce two groups of conditions (i) = (a), (b), (c), (d) and (ii) = (b'), (c) where the conditions are: (a) if $s(I)=e$ then $P(e)\subset P(I)$; (b') if $s(I)=e$ then $Z(e)\subset Z(I)$; (b) if $s(I)=e$, $E\varepsilon Z(e)$, $C\varepsilon P(I)$ then there exists $s_0\varepsilon C$ so that $s_0\cdot s(I)\varepsilon E$; (c) to any set of the class $Z(I)$ there corresponds a subset $F\varepsilon Z(I)$ such that, for any $e\subset s_0\varepsilon F$, there exists a set $E\varepsilon Z(e)$ for which $s(e)\varepsilon E$ implies $s_0-e+s(e)\varepsilon C$; (d) for every I, at least one proper subset of $S(I)$ belongs to the class $Z(I)$. The two groups of conditions correspond respectively to the Riemann and the Moore-Smith prrocesses of ordinary analysis. Indeed the latter are included as special cases, and give rise to the "weakest" of the integrals of these types. Moreover every P subject to (ii) satisfies (i), and to every P subject to (i) there is just one "associated" basis subject to (ii), in the same way that the Moore-Smith process is associated with the Riemann process. The methods are stated to apply also to abstract spaces.

The authors consider further a wider meaning of subdivision of I, and designate by interior subdivision a finite set of nonoverlapping intervals whose sum is contained in I but need not coincide with I. They introduce corresponding interior bases and integrals, each interior integral being in a certain sense "subordinate" to a corresponding P integral. The classical decomposition formula of Lebesgue is a special case of this subordination. *L. C. Young* (Cape Town).

Stone, M. H. Notes on integration. II. Proc. Nat. Acad. Sci. U. S. A. 34, 447–455 (1948).

Continuing his investigations begun in his first note on axiomatic integration [same vol., 336–342 (1948); these Rev. 10, 24], the author discusses the role in his theory of L_p spaces and of the theory of measure. He obtains the pertinent generalizations of the Hölder and Minkowski inequalities and the generalization of the Mazur theorem on the mutual homeomorphism of the L_p spaces.

An unusual aspect of the theory is the definition of measurability. A function f is called measurable if and only if mid (f, g, h) is integrable whenever g and h are integrable, where mid (a, b, c) is the intermediate one of the three numbers a, b, c. The class of measurable functions is proved to have most of its customary closure properties. It need not, however, be true that the constant function 1 is measurable. In any special case in which 1 is measurable, the theory is shown to be essentially identical with the classical measure theoretic approach to integration. As a consequence of this fact it is easy to deduce the so-called Riesz theorem concerning the representation of linear functionals on the space of continuous functions vanishing in a neighborhood of infinity in a locally compact space. *P. R. Halmos.*

Rindung, Ole. A theorem on additive set functions. Mat. Tidsskr. B. 1948, 24–28 (1948). (Danish)

A system S of sets in an abstract space is called an initial system if (1) $A_1\varepsilon S$ and $A_2\varepsilon S$ imply $A_1A_2\varepsilon S$, (2) the empty set belongs to S, (3) the relations $A=A_1+\cdots+A_n$, $A\varepsilon S$; $A_i\varepsilon S$ imply that $A=A_r$ for some $r\leqq n$. Let μ be a finite set function defined on an initial system so that $\mu(0)=0$. It is shown that μ can be extended in exactly one way to an additive function defined on the smallest field containing S.

W. Feller (Ithaca, N. Y.).

Mayrhofer, Karl. Über vollständige Masse. Monatsh. Math. 52, 217–229 (1948).

Let m be a set function such that (I) m is defined on a σ-ring of subsets of a set, (II) m is nonnegative and $m(0)=0$, (III) m is countably additive, (IV) m is complete, and (V) to every nonmeasurable subset N of each measurable set there corresponds a measurable set E of finite measure such that $N\cap E$ is nonmeasurable. The author proves that a set function satisfying (I)–(IV) satisfies (V) if and only if the class of Carathéodory measurable sets coincides with the original domain of definition of m and connects this result with the concepts of σ-finiteness and regular outer measures. *P. R. Halmos* (Chicago, Ill.).

Pettineo, B. Sulla misurabilità degli insiemi pluridimensionali. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 3, 525–530 (1947).

The following notations are used. Let S_x be the m-dimensional Euclidean space comprised of the points $x=(x_1,\cdots,x_m)$. Similarly, S_y is the n-dimensional Euclidean space comprised of the points $y=(y_1,\cdots,y_n)$. Then $S=S_x\times S_y$ is the Euclidean $(m+n)$-dimensional space. If

I is a subset of S, then I_x, I_y denote the projections of I upon the spaces S_x, S_y, respectively. For given I and $x_0 \varepsilon S_x$, $K(x_0)$ denotes the set of those points $(x, y) \varepsilon I$ for which $x = x_0$. Let T_x, T_y, T denote intervals in S_x, S_y, S, respectively. Let $\alpha(T_x)$, $\beta(T_y)$ be additive interval functions of bounded variation. They give rise, in a well-known manner, to Lebesgue measures α, β in S_x, S_y, respectively. If $T = T_x \times T_y$, then on setting $\gamma(T) = \alpha(T_x)\beta(T_y)$, we obtain in S an additive interval function of bounded variation which gives rise to a Lebesgue measure γ. Let now I be a set in S, such that the projection I_x of I is measurable relative to α. The author considers the associated set $K(x)$, defined above, and in a manner too involved for explicit statement here, defines the concept of "quasi-continuity of $K(x)$ in I_x relative to α and β." He then proves the theorem: I is measurable relative to γ if and only if $K(x)$ is quasi-continuous in I_x relative to α and β. *T. Radó.*

Pettineo, B. Sulla sommabilità delle funzioni integrabili secondo Picone-Stieltjes. II. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **3**, 512–517 (1947).

Pettineo, B. Sulla sommabilità delle funzioni integrabili secondo Picone-Stieltjes. III. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **3**, 518–520 (1947).

Pettineo, B. Sul massimo e sul minimo integrale di una funzione secondo Picone-Stieltjes. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **3**, 521–524 (1947).

These three notes continue the discussion of the Picone-Stieltjes integral defined and considered in previous notes [same vol., 230–235, 236–241 (1947); these Rev. **9**, 505]. The usual breaking up of the measure function into positive and negative parts, the corresponding divisions of sets and integrals, and their relation to the existence of the integral are developed. *T. H. Hildebrandt* (Ann Arbor, Mich.).

Dubrovskiĭ, V. M. On certain conditions of compactness. Izvestiya Akad. Nauk SSSR. Ser. Mat. **12**, 397–410 (1948). (Russian)

Let D be a closed rectangle in Euclidean n-space and let Φ be a family of completely additive set functions φ defined on the Borel or Lebesgue measurable subsets of D. The family Φ is called uniformly additive if for every sequence $\{e_n\}$ of disjoint measurable sets and every $\epsilon > 0$ there is a K such that $|\varphi(e_n + e_{n+1} + \cdots)| < \epsilon$ for all $n > K$ and all φ in Φ. The author first discusses conditions for compactness of a family Φ; a typical result is theorem II. Let each φ of Φ vanish at each point of D; then in order that each infinite subset of Φ contain a sequence convergent for every Borel set of D it is necessary and sufficient that Φ be uniformly additive. These first results are applied to families of summable functions; a typical result is theorem IV. Let N be a family of summable functions and assume uniform additivity of the family Φ of indefinite integrals of functions of N. In order that each infinite subset of N should contain a sequence pointwise convergent to a continuous limit it is necessary and sufficient that for each infinite subset N_1 of N, each $\epsilon > 0$, and each x in D there exist $\delta > 0$ and an infinite subset N_2 of N_1 (where both δ and N_2 may depend on N_1, ϵ and x) such that for every measurable $e \subseteq D$ lying in the sphere of radius ϵ about x and for every f in N_2 we have $|f(x) - (\text{meas. } e)^{-1} \int_e f(t)dt| < \epsilon$. Theorem V asserts that corresponding conditions hold for uniform convergence to a continuous limit when N_2 is no longer allowed to depend on x. *M. M. Day* (Princeton, N. J.).

Robbins, Herbert. Mixture of distributions. Ann. Math. Statistics **19**, 360–369 (1948).

By the standard method of proof of Fubini's theorem the author obtains the known generalization of that theorem to direct sum spaces and discusses certain special cases of interest in probability theory. *P. R. Halmos.*

Sengupta, H. M. On continuous independent functions. Quart. J. Math., Oxford Ser. **19**, 129–132 (1948).

Suppose that $f(x)$ and $g(x)$ are defined and continuous for $0 \leqq x \leqq 1$ and are independent functions. Then (1) if the x-set $\{x \mid g(x) = a\}$ is not empty its image under f must be the entire range of values of f. This generalizes a result of Offord [same J., Oxford Ser. **12**, 86–88 (1941); these Rev. **3**, 73] who proved that (2) if neither f nor g is identically constant each must assume all its values nondenumerably many times. The author's result is implicitly contained in a paper of Ottaviani [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **2**, 393–398 (1947); these Rev. **9**, 18]. *J. L. Doob* (Urbana, Ill.).

Behrend, F. A. Some remarks on the distribution of sequences of real numbers. Amer. J. Math. **70**, 547–554 (1948).

Let $\Phi_n(x)$ denote the distribution function of the first n numbers of a sequence of real numbers $x_1, x_2, \cdots$, which possesses an asymptotic distribution function $\Phi(x)$. The author derives sufficient conditions for the validity of the term-by-term integration $\lim_{n\to\infty} \int_{-\infty}^{\infty} f(x)d\Phi_n(x) = \int_{-\infty}^{\infty} f(x)d\Phi(x)$ for all continuous functions $f(x)$ (for which the last integral exists). It is shown that the sequence $x_1, x_2, \cdots$ can always be modified "slightly" in such a way that the asymptotic distribution function Φ is not altered and that the sufficient conditions are fulfilled. *P. Hartman* (Baltimore, Md.).

Sargent, W. L. C. On the integrability of a product. J. London Math. Soc. **23**, 28–34 (1948).

It is known that the product $f(x)k(x)$ is summable for every summable function $f(x)$ if and only if the function $k(x)$ is essentially bounded [H. Lebesgue, Ann. Fac. Sci. Univ. Toulouse (3) **1**, 25–117 (1910), in particular, pp. 38–39]. It is also known that if $f(x)$ is integrable in the Denjoy-Perron sense then it is sufficient for the integrability of $f(x)k(x)$ in this sense that $k(x)$ be equivalent to a function of bounded variation. This is a part of the general theory of Denjoy-Perron integrals. The author completes the latter result by proving the following theorem. If $k(x)$ is not equivalent to a function of bounded variation in (a, b), there is a function $f(x)$ integrable in the Cauchy sense in (a, b) and such that $f(x)k(x)$ is not integrable in the Cesàro-Perron sense of any order, nor in the general Denjoy sense in (a, b). The Cesàro-Perron integral of order zero is the Denjoy-Perron integral. *R. L. Jeffery* (Kingston, Ont.).

Chow, Shu-Er. On approximate derivatives. Bull. Amer. Math. Soc. **54**, 793–802 (1948).

Using known theorems concerning the approximate derivates of a measurable function of one variable [for example, J. C. Burkill and U. S. Haslam-Jones, Proc. London Math. Soc. (2) **32**, 346–355 (1931)] and the upper and lower measurable boundaries defined by H. Blumberg [Acta Math. **65**, 263–282 (1935)] the author obtains theorems of the Denjoy type concerning the possible dispositions of the approximate derivates A^+, A^-, A_+, A_- of an arbitrary function. He proves that, almost everywhere, one of the following three

cases must occur: (i) $A^+=A^-=+\infty$, $A_+=A_-=-\infty$; (ii) $A^+=A_-$ (finite), $A_+=-\infty$, $A^-=+\infty$; or $A_+=A^-$ (finite), $A^+=+\infty$, $A_-=-\infty$; (iii) $A^+=A_+=A^-=A_-$ (finite). He points out that case (ii) can occur in a measurable set of nonzero measure for arbitrary functions, although it cannot occur (except in a set of measure zero) for measurable functions because such functions are approximately continuous almost everywhere and therefore coincide almost everywhere with their measurable boundaries.

U. S. Haslam-Jones (Oxford).

Császár, Ákos. Sur les dérivées approximatives d'ordre supérieur. Comment. Math. Helv. 21, 253–260 (1948).

The author establishes a generalization, to approximate derivates of higher order, of the Denjoy-Khintchine theorem which states that every measurable function (*VBG*) (of generalized bounded variation) on a measurable set E is approximately differentiable almost everywhere on E. For this purpose the author first generalizes the notion of the functions (*VBG*). Let $f(x)$ be a real function defined on a linear set E, let $x_0, \cdots, x_n$ be different points of E, and let $\delta(x_0, \cdots, x_n)$ denote the diameter of the set $\{x_0, \cdots, x_n\}$. Let $Q_n(x_0, \cdots, x_n; f)$ denote the number $n!A$ if A is the coefficient of x^n in the polynomial of degree n at most, coinciding with $f(x)$ for $x=x_0, \cdots, x_n$. Moreover, set $\Delta_n(x_0, \cdots, x_n; f)=n^{-1}|Q_n(x_0, \cdots, x_n; f)|\cdot\delta(x_0, \cdots, x_n)$, which equals

$$|Q_{n-1}(x_0, \cdots, x_{n-1}; f)-Q_{n-1}(x_1, \cdots, x_n; f)| \times\delta(x_0, \cdots, x_n)/|x_0-x_n|.$$

The author says that $f(x)$ belongs to the class (R_n) on E if there exists a number M such that for each finite set of nonoverlapping intervals $I_1, \cdots, I_N$ and for $x_i' \varepsilon I_i$, one has $\sum_{i=1}^N \Delta_n(x_0', \cdots, x_n'; f)<M$. Moreover, he says that $f(x)$ belongs to the class (R_nG) on E if $E=\sum_{k=1}^\infty E_k$ and $f(x)$ is (R_n) on every E_k. The functions (R_1) and (R_1G) coincide with the functions (*VB*) and (*VBG*), respectively. Then the author proves the theorem: if the measurable function $f(x)$ is (R_nG) on the measurable set E, then the nth approximative derivative of $f(x)$ exists almost everywhere on E.

A. Rosenthal (Lafayette, Ind.).

Császár, Ákos. Les fonctions à variation bornée d'ordre supérieur. Ann. Sci. École Norm. Sup. (3) 64 (1947), 275–284 (1948).

Using the notations of the paper reviewed above, the author now gives a generalization, to higher derivatives, of the Denjoy-Lusin theorem which states that every function (*VBG**) (of generalized bounded variation in the restricted sense) on a set E is differentiable almost everywhere, and, simultaneously, he gives an extension of a necessary and sufficient condition of Denjoy for a function to be differentiable almost everywhere on E. The limit $f^{[n]}(a)=\lim_{x_i\to a} Q_n(x_0, \cdots, x_n; f)$ is called a strong derivative of order n of the function $f(x)$ at a, while $f^*(a)=\lim_{x_i\to a} Q_n(x_0, \cdots, x_n; f)$ with $\min x_i \leqq a \leqq \max x_i$ is called a weak derivative of order n of $f(x)$ at a. Moreover, $f(x)$ is said to belong to the class (R_n^*) on E if there exists an M such that for every finite set of nonoverlapping intervals (a_ν, b_ν) with $a_\nu \varepsilon E$, $b_\nu \varepsilon E$ one has $\sum_\nu \Omega_n(a_\nu, b_\nu; f)<M$, where $\Omega_n(a, b; f)=\lim\sup_{a\leqq x_i\leqq b} \Delta_n(x_0, \cdots, x_n; f)$. Analogously, $f(x)$ is said to belong to the class (S_n^*) on E if for every $\epsilon>0$ the set E can be covered by finitely many intervals $[a_\nu, b_\nu]$ such that $\sum_\nu \Omega_n(a_\nu, b_\nu; f)\cdot(b_\nu-a_\nu)<\epsilon$. Finally, the author says that $f(x)$ belongs to the class (R_nG^*) or (S_nG^*) on E if $E=\sum_{k=1}^\infty E_k$ and $f(x)$ is (R_n^*) or (S_n^*), respectively, on each E_k. The functions (R_1G^*) coincide with the functions (*VBG**). Then the author proves the following two theorems. In order that $f^n(x)$ or $f^{[n]}(x)$ exist almost everywhere on E, it is necessary and sufficient that E be the sum of a set of measure zero and of a set on which $f(x)$ is (R_nG^*) or $(S_{n+1}G^*)$, respectively.

A. Rosenthal.

Cesari, L. Proprietà tangenziali delle superficie continue. Comment. Math. Helv. 22 (1949), 1–16 (1948).

Let a continuous surface S be given by equations $x=x(u, v)$, $y=y(u, v)$, $z=z(u, v)$, $(u, v)\varepsilon Q$, where Q is the unit square $0\leqq u\leqq 1$, $0\leqq v\leqq 1$. Let us assume that the Lebesgue area $L(S)$ of S is finite. Consider the projection transformations: Φ_1: $y=y(u, v)$, $z=z(u, v)$; Φ_2: $z=z(u, v)$, $x=x(u, v)$; Φ_3: $x=x(u, v)$, $y=y(u, v)$. As a consequence of the assumption $L(S)<\infty$, there exists then a generalized Jacobian $H_r(u, v)$, in the sense of the author, for the transformation Φ_r almost everywhere in Q, for $r=1, 2, 3$. Let us now introduce new Cartesian coordinate axes x^*, y^*, z^*, oriented as the old axes x, y, z. Let Φ_r^* be the new projection transformations, with corresponding generalized Jacobians H_r^*, $r=1, 2, 3$. The author proves that the new Jacobians H_r^* are related to the old Jacobians H_r by the same transformation formulas as those that are familiar for ordinary Jacobians in smooth cases. From this result, combined with some of his previous results, the author derives then the following main theorem. For almost every point $P=(u, v)\varepsilon Q$, it is possible to introduce new Cartesian axes x^*, y^*, z^* in such a manner that the following is true: (i) $H_2^*(P)=0$, $H_3^*(P)=0$; (ii) $H_1^*(P)=\lim L(q)/|q|$, where $L(q)$ is the Lebesgue area of the portion of the surface S corresponding to a square $q\subset Q$, and the limit process is relative to squares q such that $P\varepsilon q$ and $|q|\to 0$. This result may be interpreted to mean, as the author notes, that in a certain statistical sense the surface S behaves infinitesimally as a plane if $L(S)<\infty$.

T. Radó (Columbus, Ohio).

Young, L. C. Some applications of the Dirichlet integral to the theory of surfaces. Trans. Amer. Math. Soc. 64, 317–335 (1948).

A set of curves with uniformly bounded lengths and end points is compact, but the corresponding statement is not valid for surfaces. The main purpose of this paper is to study modifications of concepts which permit a similar result to be established for surfaces. The following definitions are made. Let S be a surface defined parametrically on the unit square R of the $w=u+iv$ plane by means of a continuous vector function $x=f(w)$. Then S is termed a Dirichlet surface, and $x=f(w)$ is called a Dirichlet representation for S, if f is absolutely continuous on almost all parallels to the axes of u and v, and the Dirichlet integral of f over R, namely $\iint_R \frac{1}{2}\{(f_u)^2+(f_v)^2\}dudv$, is finite. The surface represented by $x=f(w)$ is said to have a split boundary-curve if the perimeter of R may be expressed as the sum of two arcs W' and W'' having only their extremities in common, so that f has a positive oscillation over both W' and W'', and the extremities of W' and W'' belong to a continuum of constancy of f. A sequence of surfaces is said to possess an asymptotic split boundary-curve if it contains a subsequence of surfaces S_n having boundary curves C_n with Fréchet limit C for which the following holds. A double point divides C into two closed subcurves C' and C'', neither of which reduces to a point, which are expressible as Fréchet limits of two corresponding curves

C_n' and C_n'' on the surface S_n, where C_n' and C_n'' consist respectively of two complementary arcs of C_n together with a common arc K_n whose diameter tends to zero with n^{-1}. A set of surfaces is termed quasi-compact if for each positive number ϵ it can be transformed into a compact set of surfaces by replacing certain portions of each surface which are the images of open subdomains of the parameter domain by new portions whose total area is less than ϵ. The following theorems are established. A sequence of Dirichlet surfaces whose intrinsic areas [L. C. Young, forthcoming paper] are bounded and whose boundary-curves tend to a closed curve which is no asymptotic-split boundary curve, or in particular tend to a simple closed curve, is quasi-compact. A sequence of Dirichlet surfaces whose classical areas are bounded and whose boundary-curves tend to a closed curve which is no asymptotic split boundary-curve is quasi-compact. Finally, a sequence of surfaces whose Lebesgue-Fréchet areas are bounded and whose boundary curves are unsplit and tend to a closed curve which is no asymptotic split boundary curve is quasi-compact.

P. V. Reichelderfer (Columbus, Ohio).

Theory of Functions of Complex Variables

Ghika, Al. Sur le prolongement d'une fonction analytique sur un domaine riemannien donné. Acad. Roum. Bull. Sect. Sci. **26**, 155–161 (1946).

The author generalizes to a domain D on a Riemann surface results he had obtained previously [cf. C. R. Acad. Sci. Paris **188**, 607–609 (1929)] for schlicht domains. He is able to state a condition for an analytic function $f(z)$ given by its Taylor series at a point $a \in D$ which ensures the possibility of continuing $f(z)$ throughout D and of assuming square integrable boundary values on the boundary of D. This condition is of the form

$$\sum_{n=0}^{\infty}\left|\sum_{k=0}^{n}\mu_{nk}f^{(k)}(a)/k!\right|^2<\infty,$$

where μ_{nk} depend only on D and a. A formally similar condition guarantees the same for a function $f(z)$ given by its values $f(a_n)$ on a sequence of points having a limit point in D.

František Wolf (Berkeley, Calif.).

Peilegrino, Franco. Una condizione necessaria e sufficiente perchè una serie di potenze abbia sulla circonferenza di convergenza un solo polo multiplo. Pont. Acad. Sci. Acta **6**, 115–123 (1942).

The function $\sum_{n=0}^{\infty}a_nz^n$ has as the only singularity on its circle of convergence a pole of order h at $z=\alpha$ if and only if $a_{n+1}/a_n \sim 1/\alpha$ and

$$\limsup\left|1-\binom{h}{1}\frac{a_{n-1}}{a_n}+\binom{h}{2}\frac{a_{n-2}}{a_n}+\cdots+(-)^h\binom{h}{h}\frac{a_{n-h}}{a_n}\right|^{1/n}<1/|\alpha|.$$

This result is proved by considering the function $(z-\alpha)^h\sum a_nz^n$ and generalises classical results of Hadamard for $h=1$. It is also shown that $\lim n(a_n/a_{n-1}-\alpha)=1-h$ and gap theorems due to Mandelbrojt [Ann. Sci. École Norm. Sup. (3) **4,0** 413–462 (1923)] are deduced. *A. J. Macintyre.*

Roussel, André. Sur les développements tayloriens d'une fonction. C. R. Acad. Sci. Paris **227**, 500–502 (1948).

Let $g(x, h)$ be holomorphic in the circles $|x-a|\leqq R$, $|h|\leqq R'$, with $g(x, 0)=0$. A sequence of functions $\{g_n(x, h)\}$ are determined from $g(x, h)$ as in an earlier paper [same C. R. **225**, 23–24 (1947); these Rev. **9**, 21], where it was shown that the series $\sum_{-\infty}^{\infty}g_n(x, h)$ converges absolutely and uniformly for $|x-a|+|h|\leqq\rho<\min(R, R')$ to $f(x+h)-f(x)$, with

$$f(x)=g(a, x-a)+\int_0^{x-a}g_z'(x-t, t)dt.$$

In particular, $f(x)-f(a)=g(a, x-a)+\sum_1^{\infty}g_n(a, x-a)$.

In this paper, the author considers the problem of finding $g(x, h)$ for a given function $f(x)$ so that

$$g(a, x-a)+\int_0^{x-a}g_z'(x-t, t)dt=f(x)-f(a).$$

For an arbitrary kernel $K(s, t)$ holomorphic in the neighborhood of $s=0$, $t=0$, $u(s)$ is found as the solution of

$$u(s)-\int_0^s K(s, t)u(t)dt=f(a+s)-f(a),$$

whence

$$g(x, h)=-u(h)\cdot\int_0^{x-a+h}K(\sigma, h)d\sigma+u(h).$$

The choice $K(s, t)=-f''(a+s-t)/f'(a)$, $f'(a)\neq 0$, leads to $u(s)=f'(a)s$, $g(x, y)=hf'(x)$ and the classical Taylor expansion for $f(x)-f(a)$ about the point a. *M. S. Robertson.*

van de Putte, J. G. Inversion and the function $w=\dfrac{az+b}{cz+d}$. Simon Stevin **26**, 15–24 (1948). (Dutch)

Clément, Lucette. Étude de la surface de Riemann de $f(z)=e^{hz}\dfrac{e^z-1}{z}$, $h>0$. C. R. Acad. Sci. Paris **227**, 256–257 (1948).

The title contains a misprint. The function discussed is $f(z)=e^{hz}(e^z-1)/z$. The curves $\arg f(z)=$ constant and the distribution of winding points are described.

A. J. Macintyre (Aberdeen).

Geronimus, Ya. L. On certain inequalities in the theory of functions of Carathéodory and Schur type. Doklady Akad. Nauk SSSR (N.S.) **60**, 953–956 (1948). (Russian)

The author recently improved an inequality of I. Schur from $|f(z)|<\{(G-1)/G\}^{\frac{1}{2}}$ to the best possible form $|f(z)|\leqq(G-1)/(G+1)$ [I. Schur, J. Reine Angew. Math. **147**, 205–232 (1917); **148**, 122–145 (1918), in particular, p. 143; Geronimus, Rec. Math. [Mat. Sbornik] N.S. **15(57)**, 99–130 (1940); these Rev. **7**, 63]. The present paper gives a new and elementary proof which proceeds as follows. If $f(z)$ is regular and $|f(z)|<1$ for $|z|<1$ a sequence of similar functions and a sequence of parameters is defined by Schur's algorithm: $f_{n+1}(z)=\{f_n(z)-\gamma_n\}/z\{1-\bar{\gamma}_nf(z)\}$, $\gamma_n=f_n(0)$, $f_0(z)\equiv f(z)$. The inequality applies if $G=\prod_0^{\infty}(1+|\gamma_n|)/(1-|\gamma_n|)$ converges. It is proved first for the special rational function for which $\gamma_0, \cdots, \gamma_n$ take the given values and $f_n(z)\equiv\gamma_n$ by an induction from $f_{n+1}(z)$ to $f_n(z)$. The general inequality easily follows. The parallel results for functions regular and satisfying $\Re\{f(z)\}>0$ for $|z|<1$ are derived.

A. J. Macintyre (Aberdeen).

Rabinovič, Yu. L. Inversion formulas for two kinds of Laplace transform. Doklady Akad. Nauk SSSR (N.S.) **60**, 969–972 (1948). (Russian)

The two forms of transform are substantially those usually described as the Laplace transform and its inverse and are

discussed for functions analytic near infinity [cf. A. Pfluger, Comment. Math. Helv. 8, 89–129 (1935)]. By the introduction of a very natural generalisation, however, the two forms of transform are exhibited in a new relationship and a case where the inverse is not unique is included. The theory is developed very simply from Cauchy's theorem.

The paper may be summarised as follows. If $f(z)$ is regular (not necessarily one-valued) for $|z|>\rho$ and satisfies $|f(z)|<Ke^{\lambda|z|}$ then the integral (the "first kind of Laplace integral") $g(z)=\int_{z_0}^{\infty}f(\zeta)e^{-z\zeta}d\zeta$ $(|z_0|>\rho)$ defines $g(z)$ in the half plane $\Re(ze^{i\theta})>\lambda$ provided the contour extends to ∞ finally along the radius $\arg\zeta=\theta$. Analytic continuation of $g(z)$ throughout $|z|>\lambda$ is effected by rotating this radius and $g(z)$ is, like $f(z)$, of exponential type near infinity. The difference between two adjacent branches of $g(z)$ for $\Re(ze^{i\theta})>\lambda$ is given by $h(z)=\int_{(0)}^{\infty}f(\zeta)e^{-z\zeta}d\zeta$, where the contour encloses the circle $|\zeta|=\rho$ and extends to infinity in the direction θ at both ends. If the "first form of transform" is applied in turn to $h(z)$ and the order of integration in the double integral is reversed there results the integral $\int_{(0)}^{\infty}f(\zeta)\exp\{-z_0(x+\zeta)\}(x+\zeta)^{-1}d\zeta$ which, from the calculus of residues, is easily seen to differ from $2\pi if(-x)$ by an integral function. For the class of functions envisaged the first transform is thus essentially the inverse of the second apart from an "error" which is an integral function. Conversely the second transform is the inverse of the first in the same sense. This proof is a little more complicated. Consider $\int_{(0)}^{\infty}g(z)e^{-xz}dz$ for positive x and distort the contour into the pair of lines $\arg(z+A)=\pm\delta$ ($A>0$ large, $\delta>0$ small). On the arm $\arg(z+A)=+\delta$ $(-\delta)$ use the integral defining $g(z)$ with its contour extending to infinity in the direction $\frac{1}{2}\pi+\frac{1}{2}\delta$ $(\frac{3}{2}\pi-\frac{1}{2}\delta)$. The argument then proceeds much as before.

The paper also discusses the relation between the growth diagram of each function and the convex envelope of the singularities of the other. The results and methods are similar to those given by Pfluger. *A. J. Macintyre.*

Leja, F. Sur le domaine de convergence des séries de polynômes homogènes à deux variables. Ann. Acad. Polonaise Sci. Tech. Varsovie 7, 9 pp. (1945).

The author restates and supplements some of his previous results about convergence of power series in several variables [Rend. Circ. Mat. Palermo 56, 419–445 (1932); Ann. Soc. Polon. Math. 12, 29–34 (1934)]. In Euclidean E_k take a sequence of homogenous polynomials $P_n(x)$, $n=0,1,\cdots$, and a point set D, and assume that on D either the series $\sum_0^{\infty}P_n(x)$ converges or the sequence is bounded. Assume the latter; thus $|P_n(x)|\leq N$ for x in D. The question then is: for what D will the series converge absolutely in a neighborhood of the origin? To answer it, take n points $\xi^1,\cdots,\xi^n$ in D, set up the Lagrange interpolation formula $P_n(x)=\sum_{j=1}^{n}P_n(\xi^j)\Delta_n^j(x)/\Delta_n^j(\xi^j)$ and estimate thus: $M^{-1}P_n(x)\leq\sum_{j=1}^{n}|\Delta_n^j(x)|/|\Delta_n^j(\xi^j)|$. The decisive thing then is to keep the nth root of $\sum_{j=1}^{n}|\Delta_n^j(\xi^j)|^{-1}$ below a fixed bound, or, what is the same, to keep its reciprocal away from 0. This then is expressible as a requirement that a certain "capacity" of the point set D relative to the origin, and definable for any E_k, has a strictly positive value. The author's investigations have some connection with (although they are by no means a substitute for) Hartogs' theorem on the continuity of a function in several variables which is assumed to be analytic in each variable separately.

S. Bochner (Princeton, N. J.).

Martinelli, Enzo. Formule integrali e topologia nella teoria delle funzioni di più variabili complesse. Pont. Acad. Sci. Acta 9, 235–250 (1945).

Verf. stellt neue Verallgemeinerungen des Cauchyschen Satzes und der Cauchyschen Integralformel der klassischen Theorie für analytische Funktionen von n komplexen Veränderlichen auf, wobei die Integrationsfläche von beliebiger Dimension $n+l$ zwischen n und $2n-1$ ist. Eine ausführliche Darstellung seiner Resultate will Verf. in einer nächsten Arbeit geben. *P. Thullen* (Bogotá).

Casulleras, Juan. On the representation of a complex E_n by means of a real E_{2n}. Revista Mat. Hisp.-Amer. (4) 8, 51–56 (1948). (Spanish)

Elementare Darstellung der unendlichfernen Punkte eines Raumes von n komplexen Veränderlichen in einem reellen $2n$-dimensionalen Raum [Erweiterung einer von Teixidor für den Fall von zwei komplexen Veränderlichen angegebenen Methode: Siehe Revista Mat. Hisp.-Amer. (4) 7, 173–177 (1947); diese Rev. 9, 278]. *P. Thullen* (Bogotá).

Petersson, Hans. Über den Bereich absoluter Konvergenz der Poincaréschen Reihen. Acta Math. 80, 23–63 (1948).

This discussion of the convergence of the Poincaré series is of special interest because the methods used are of an elementary nature. The proofs are self contained and complete and they treat the parabolic, hyperbolic and elliptic cases in a uniform way. *H. S. Zuckerman.*

Pic, G. Sur quelques propriétés des groupes discontinus et finis de substitutions linéaires. Mathematica, Timişoara 23, 131–138 (1948).

Remarques élémentaires concernant les groupes discontinus et finis de substitutions linéaires de plusieurs variables. *P. J. Myrberg* (Helsinski).

Fueter, Rudolf. Über die Funktionentheorie in einer hyperkomplexen Algebra. Elemente der Math. 3, 89–94 (1948).

Expository article.

Theory of Series

Sibirani, Filippo. Inversione di alcune funzioni. Mem. Accad. Sci. Ist. Bologna. Cl. Sci. Fis. (10) 2, 65–74 (1946).

For some given functions $f(x)$ analytic at $x=0$, the coefficients a_n in the expansion $x=\sum a_n y^n$ of the inverse of the function $y=xf(x)$ are obtained by writing $x=\sum a_n[xf(x)]^n$, rearranging the right side, and equating coefficients of powers of x. Examples: $f(x)=\exp x^p$; $(1+x)^{-k}$; $1/(1+x^k)$; $\cos x$. *R. P. Agnew* (Ithaca, N. Y.).

Sibirani, Filippo. Sommazione di serie. Mem. Accad. Sci. Ist. Bologna. Cl. Sci. Fis. (10) 3, 69–77 (1947).

It is a familiar practice to evaluate convergent numerical series by the following method. The value of the series $\sum_{n=1}^{\infty}n/(n-1)!$ is $f(1)$, where

$$f(x)=\sum nx^n/(n-1)!=x(d/dx)\sum x^n/(n-1)!$$
$$=x(d/dx)xe^x=x^2e^x+xe^x$$

The author gives formulas which lead efficiently to many results obtainable by such methods. Let I_1 denote the operator $x(d/dx)$, and let I_n denote its nth iterate. The numbers $g_{n,k}$ defined by the identities $I_n\varphi(x)=\sum_{k=1}^{n}g_{n,k}x^k\varphi^{(k)}(x)$ sat-

isfy a recursion formula which enables one to write the numbers almost as quickly as one writes the binomial coefficients in the Pascal triangle. The coefficients in the expansion in powers of x of $f(\varphi(x))$ are written as expressions involving the numbers $g_{n,k}$. By specializing the functions f and φ, and using the values of the numbers $g_{n,k}$, many numerical series are evaluated. For example,

$$\sum_{n=1}^{\infty}\frac{n^7}{(n-1)!}=4140e;\quad \sum_{n=1}^{\infty}\frac{(2n)^6}{(2n-1)!}=443\sinh 1+434\cosh 1.$$

R. P. Agnew (Ithaca, N. Y.).

Pettineo, Benedetto. Sulle serie numeriche $\sum u_n f(n)$, dove la successione u_n è monotona e la funzione $f(x)$ è periodica. Ann. Mat. Pura Appl. (4) **26**, 119–140 (1947).

Let $\sum u_n$ be a series whose terms form a monotone decreasing sequence of positive numbers. Let $f(x)$ be a function, with period θ, having a Riemann integral over a period equal to μ. If $\mu\neq 0$ and θ is irrational, then $\sum u_n f(n)$ diverges with the same rapidity as $\sum \mu u_n$. If $\theta=h/K$, where h and K are positive integers, and if $\lambda=h^{-1}[f(1)+\cdots+f(h)]\neq 0$, then $\sum u_n f(n)$ diverges as $\sum \lambda u_n$. *R. P. Agnew.*

Prachar, K. Zur Geometrie der Reihen. Monatsh. Math. **52**, 255–259 (1948).

H. Hornich [Monatsh. Math. Phys. **45**, 432–434 (1937)] proved the following fact concerning series $\sum a_n$ of complex terms such that $a_n\to 0$ and $\sum|a_n|=\infty$. If k is an integer, $k\geqq 3$, if ζ is a primitive kth root of unity, then to each complex number A corresponds a sequence $p_1, p_2, \cdots$ of positive integers such that the series $\sum a_n\zeta^{p_n}$ converges to A. Assuming the stronger hypothesis that $a_n\to 0$ and $|a_1+a_2+\cdots+a_n|\to\infty$, the author shows that the sequence $p_1, p_2, \cdots$ can be chosen such that, for each n, p_{n+1} is either p_n or p_n+1. *R. P. Agnew* (Ithaca, N. Y.).

Bosanquet, L. S. Note on convergence and summability factors. II. Proc. London Math. Soc. (2) **50**, 295–304 (1948).

[For part I, cf. J. London Math. Soc. **21**, 11–15 (1946); these Rev. **8**, 259.] Let r and s be integers such that $-1\leqq r\leqq s$, and let p be a real number. In order that a sequence $\epsilon_0, \epsilon_1, \epsilon_2, \cdots$ be such that the series $\sum a_n\epsilon_n$ is summable C_r whenever

$$S_n^{(s)}=\sum_{k=0}^{n}\binom{s-1}{n-k}s_k=\sum_{k=0}^{n}\binom{s-1}{n-k}\sum_{j=0}^{k}a_j=O(n^{s+p}),$$

it is necessary and sufficient that (i) $\epsilon_n=o(n^{r-s-p})$ and (ii) $\sum(n+1)^{s+p}|\Delta^{s+1}\epsilon_n|<\infty$. If $r>s\geqq 0$, the conditions are the same as when $r=s$; but if $r>s=-1$, the first condition must be omitted. The same holds with O and o interchanged. *R. P. Agnew* (Ithaca, N. Y.).

Bosanquet, L. S. On convergence and summability factors in a Dirichlet series. II. J. London Math. Soc. **23**, 35–38 (1948).

[Cf. même J. **22** (1947), 190–195 (1948); ces Rev. **9**, 581.] L'auteur démontre le théorème suivant. Soit $\varphi(t)$ une fonction à variation bornée dans chaque intervalle $(0, x)$ et soit pour un $\kappa\geqq 0$, un $p>-\kappa$ et $\delta>0$, $\int_1^u d\varphi(u)=O(x^p)(C,\kappa)$ pour $x\to\infty$. Alors l'intégrale $\int_1^\infty u^{-p-1}d\varphi(u)$ est sommable $|C,\kappa+1|$. On en déduit, par exemple, que l'abscisse de sommabilité $|R,l,\kappa+1|$ de la série de Dirichlet $\sum a_n l_n^{-s}$ ne surpasse pas l'abscisse de sommabilité (R,l,κ).

N. Obrechkoff (Sofia).

***Teghem, Jean. Sur des Procédés de Sommation Issus de la Transformation d'Euler.** Thesis, Université Libre de Bruxelles, 1946. 88 pp.

This thesis is dated 1946 on the title page, but a footnote on page 50 shows that it was printed in 1948. It gives a full account of results summarized earlier [Bulletin Soc. Roy. Sci. Liége **14**, 366–376 (1945); these Rev. **8**, 511].

R. P. Agnew (Ithaca, N. Y.).

Darevskiĭ, V. M. Conditions for consistency of Toeplitz methods. Izvestiya Akad. Nauk SSSR. Ser. Mat. **12**, 379–396 (1948). (Russian)

Let A and B be regular sequence-to-sequence matrix methods of summability of the Silverman-Toeplitz type. Let A be row-finite, and let $X_1, X_2, \cdots$ be a sequence of positive numbers such that $x_n=O(X_n)$ whenever the sequence $x_1, x_2, \cdots$ is summable A. Conditions, too complicated to be written and explained here, are given which imply that A and B are consistent. That the theorem is nontrivial is shown by exhibition of methods A and B such that the conditions are satisfied, neither of A and B includes the other, and some divergent sequences are evaluable by both A and B. The results and methods of proof are similar to those of a previous paper [Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **11**, 3–32 (1947); these Rev. **8**, 510] where the author treated summability of bounded sequences. *R. P. Agnew* (Ithaca, N. Y.).

Dieudonné, Jean. Sur la méthode du col. Bol. Soc. Mat. São Paulo **2**, 7–34 (1947).

The author discusses the asymptotic behavior of integrals of the type $\int_0^\infty \exp(\pm xt+x^n\Phi(t))dt$ as $x\to\infty$ or $x\to 0$, where Φ is a logarithmico-exponential function (in the sense of Hardy). *P. Hartman* (Baltimore, Md.).

van der Corput, J. G. On the method of critical points. I. Nederl. Akad. Wetensch., Proc. **51**, 650–658 = Indagationes Math. **10**, 201–209 (1948).

Expository lecture on an extension of the method of stationary phases for obtaining asymptotic expansions.

P. Hartman (Baltimore, Md.).

Pitt, H. R. A note on some elementary Tauberian theorems. Quart. J. Math., Oxford Ser. **19**, 177–180 (1948).

If $k(u, y)$ and $s(y)$ are suitably restricted functions, information about $s(y)$ is obtained from information about the transform $g(u)=\int_{-\infty}^{\infty}k(u, y)s(y)dy$. Let $\int_{-\infty}^{\infty}k(u, y)dy=1$, $-\infty<u<\infty$, and

$$\lim_{u\to\infty}\int_{-\infty}^{Y}k(u, y)dy=0,\qquad -\infty<Y<\infty.$$

The first theorem shows that if $k(u, y)\geqq 0$, if $s(x)$ is bounded, if $s(x)$ satisfies one of the general Tauberian conditions of the author's paper [Proc. London Math. Soc. (2) **44**, 243–288 (1938)], and if $\Lambda(\epsilon)>\frac{1}{2}$, where $\Lambda(\epsilon)$ is the limit of the inferior limit of the least upper bound of an integral involving $k(u, y)$ and parameters in the Tauberian condition, then $\limsup|s(x)|\leqq[\epsilon+\limsup|g(x)|]/[2\Lambda(\epsilon)-1]$ and, if $\Lambda(+0)>\frac{1}{2}$, $\limsup|s(x)|\leqq[\limsup|g(x)|]/[2\Lambda(+0)-1]$. The second theorem shows that if the same hypotheses hold and $s(x)$ satisfies a stronger Tauberian condition, then the core of $s(x)$ (the least closed convex set containing the set of limit points of $s(x)$) coincides with the core of $g(x)$. The third and final theorem replaces the hypothesis that $k(u, y)\geqq 0$ by the hypothesis that $\int|k(u, y)|dy\leqq M$, $-\infty<u<\infty$, and gives conditions on $s(x)$ and $k(u, y)$ suffi-

cient to ensure that each limit point of $s(x)$ is also a limit point of $g(u)$.

A remark in the introduction must be amended. It is said that if $k(x)$ is integrable over $-\infty<x<\infty$, if $s(x)$ is bounded and differentiable and such that $s'(y)\to 0$ as $y\to\infty$, and if (*) $\lim_{y\to\infty}\int_{-\infty}^{\infty}k(u-y)s(y)dy$ exists, then $\lim_{y\to\infty}s(y)$ exists, "whatever the form of $k(x)$." It must be assumed that $\int_{-\infty}^{\infty}k(x)dx\neq 0$ because if the integral is 0 then the limit in (*) is zero for all bounded differentiable functions $s(y)$ which oscillate slowly, like sin log y, as $y\to\infty$. *R. P. Agnew.*

Fourier Series and Generalizations, Integral Transforms

Zamansky, Marc. Sur l'approximation des fonctions continues. C. R. Acad. Sci. Paris **227**, 463–464 (1948).

Let $f(x)$ be a continuous function of period 2π and let $\sigma_n(x)$ be the $(C, 1)$ means of its Fourier series. S. Bernstein [Acad. Roy. Belgique. Cl. Sci. Mém. Coll. in 8°. (2) **4**, no. 1 (1912)] proved that $\sigma_n(x)-f(x)=O(n^{-\alpha})$, $0<\alpha<1$, uniformly in x, if and only if $f\in\mathrm{Lip}\,\alpha$. The result is false for $\alpha=1$. The author states the following theorem. A necessary and sufficient condition for $\sigma_n(x)-f(x)=O(1/n)$ is that $\int_\epsilon^\pi t^{-2}\{f(x+t)+f(x-t)-2f(x)\}dt$ be uniformly bounded in x and $\epsilon>0$. *A. Zygmund* (Chicago, Ill.).

Chandrasekharan, K. On the summation of multiple Fourier series. I. Proc. London Math. Soc. (2) **50**, 210–222 (1948).

Let $f(x)=f(x_1,\cdots,x_k)$ be a function of k variables $(x_1,\cdots,x_k)$, of period 2π in each, and of the class L. Let $\sum a_{n_1\cdots n_k}e^{i(n_1x_1+\cdots+n_kx_k)}$ be the Fourier series of f, and

$$S_R(x)=\sum_{\gamma\leq R}a_{n_1\cdots n_k}e^{i(n_1x_1+\cdots+n_kx_k)},\qquad \gamma^2=n_1^2+\cdots+n_k^2,$$

the "spherical" partial sums of the series. Following an idea of Bochner [Trans. Amer. Math. Soc. **40**, 175–207 (1936)] the author considers the series as a simple series

$$\sum_{j=0}^{\infty}\sum_{\gamma=R_j}a_{n_1\cdots n_k}e^{i(n_1x_1+\cdots+n_kx_k)},$$

where R_j is the sequence of all integers that can be represented as a sum of k squares, and investigates its summability (R, γ^2, δ), that is, the limit, for $R\to\infty$, of

$$S_R^\delta=\sum_{\gamma\leq R}(1-\gamma^2/R^2)a_{n_1\cdots n_k}e^{i(n_1x_1+\cdots+n_kx_k)}.$$

This leads to considering the averages

$$f(x,t)=(2\pi)^{-k}\int_\sigma f(x_1+t\xi_1,\cdots,x_k+t\xi_k)d\sigma_\xi,$$

where σ denotes the unit sphere $\xi_1^2+\cdots+\xi_k^2=1$ and $d\sigma_\xi$ its $(k-1)$-dimensional volume-element, and the averages

$$f_p(x,t)=\frac{2}{B(p,\frac{1}{2}k)t^{p+\frac{1}{2}k-2}}\int_0^t(t^2-s^2)^{p-1}s^{k-1}f(x,s)ds$$

of $f(x,s)$ with respect to s. It is shown that a necessary and sufficient condition for the Fourier series of f to be summable at a point $x=(x_1,\cdots,x_k)$ by some method (R,γ^2,δ) is that $\lim_{t\to 0}f_p(x,t)$ exist for some p. Relations between γ and p are given. [For $k=1$, see Hardy and Littlewood, Math. Z. **19**, 67–96 (1923); Bosanquet, same Proc. (2) **31**, 144–164 (1930); **37**, 17–32 (1934).] *A. Zygmund.*

Chandrasekharan, K. On the summation of multiple Fourier series. II. Proc. London Math. Soc. (2) **50**, 223–229 (1948).

The results of part I [cf. the preceding review] are extended to absolute summability $|R, \nu^2, \delta|$. It is shown that a necessary and sufficient condition that $f_p(x,t)$ be, for a fixed x, of bounded variation in $0<t<\infty$ for some p is that $S_R^\delta(x)$ be of bounded variation in $0<R<\infty$ for some δ. The relations between p and ν are given. *A. Zygmund.*

Alexits, Georges. Sur la convergence des séries lacunaires. Acta Univ. Szeged. Sect. Sci. Math. **11**, 251–253 (1948).

Let $\varphi_0(x), \varphi_1(x), \cdots$ be an orthonormal system in an interval (a, b), and let S denote a series $\sum c_n\varphi_n(x)$. Let λ_n be a nondecreasing sequence of positive numbers. Then S is called λ_n-lacunary if the number of the indices m, situated in the interval $(2^n, 2^{n+1})$ and such that $c_m\neq 0$ is $O(\lambda_{2^n})$. It is shown that, if S is λ_n-lacunary, if it is summable $(C, 1)$ almost everywhere in (a, b), and if $\sum\lambda_n c_n^2<\infty$, then S converges almost everywhere in (a, b). *A. Zygmund.*

Rényi, Alfred. Remarque à la note précédente. Acta Univ. Szeged. Sect. Sci. Math. **11**, 253 (1948).

A modification of the result of Alexits reviewed above. *A. Zygmund* (Chicago, Ill.).

Chen, Kien-Kwong. An extension of Parseval's formula in the theory of orthogonal functions. Ann. of Math. (2) **49**, 511–514 (1948).

Suppose that the $\varphi_n(x)$ form a normal orthogonal system on the interval $[0, 1]$ and that $|\varphi_n(x)|\leq M$. It is proved that, when f is L^p, where $1<p\leq 2$, then

$$\left\{\int_0^1\left|\sum_{n=1}^{n(x)}c_n\varphi_n(x)\right|^q dx\right\}^{1/q}\leq C(\log n)^{2/q}M^{(2/p)-1}\left\{\sum_{n=1}^{n}|c_n|^p\right\}^{1/p}.$$

Here C is an absolute constant, $p^{-1}+q^{-1}=1$, and $0\leq n(x)\leq n$. This is an extension to the Riesz-Hausdorff type of an inequality of Menchoff ($p=q=2$) [Fund. Math. **4**, 82–105 (1923)]. It is derived from this by a convexity theorem on linear functionals. *W. W. Rogosinski.*

Orlicz, W. Une généralisation d'un théorème de Cantor-Lebesgue. Ann. Soc. Polon. Math. **21**, 38–45 (1948).

A set B of real numbers is called a rational basis if every real number x has a (not necessarily unique) representation $x=\sum_1^k r_j x_j$, with $x_j\in B$ and r_j rational; and the real functions $f_j(x)$, $j=1,\cdots,n$, are said to be linearly independent in the strong sense on the (finite or infinite) interval $[a, b]$ if for every set $p_1,\cdots,p_n$ of real numbers, not all zero, $\sum_1^n p_j f_j(x)$ is zero on at most a denumerable set in $[a, b]$. The following result is proved. Let $f_j(x)$, $j=1,\cdots,k$, be real functions on $(-\infty,\infty)$, of period l, linearly independent in the strong sense on $[0, l]$, and discontinuous in at most a denumerable set. If for every x of a rational basis B, sequence (1) $\sum_{j=1}^k a_n^{(j)}f_j(\omega_n x+\theta_n)$ converges to zero [sequence (1) is bounded], then the sequences (2) $a_n^{(j)}$ $(j=1,\cdots,k)$ converge to zero [the sequences (2) are bounded]. Here θ_n is any sequence of numbers and ω_n any sequence for which $\omega_n\neq 0$ and $\omega_n\to+\infty$. A corollary is the following generalization of the Cantor-Lebesgue theorem. Let $f(x)$, $g(x)$ be real functions on $(-\infty,\infty)$, of period 1, linearly independent in the strong sense on $[0, 1]$, and discontinuous in at most a denumerable set. If $\sum_0^\infty\{a_n f(nx)+b_n g(nx)\}$ converges for each x of a rational basis B, then $a_n\to 0$, $b_n\to 0$. *I. M. Sheffer* (State College, Pa.).

Krein, S., and Levin, B. On a problem of I. P. Natanson. Uspehi Matem. Nauk (N.S.) **3**, no. 3(25), 183–186 (1948). (Russian)

Let $K(h, x)$ and $S(h, x)$ be two kernels defined and continuous for $0<x\leqq h\leqq 1$. The authors discuss the problem of conditions under which

$$(*) \qquad \lim_{h\to 0}\int_0^h K(h, x)f(x)dx$$

implies

$$(*') \qquad \lim_{h\to 0}\int_0^h S(h, x)f(x)dx,$$

where $f(x)$ is integrable over $(0, 1)$. They show that, if the partial derivatives $K_x'(h, x)$ and $S_x'(h, x)$ exist and are continuous for $0<x\leqq h\leqq 1$, if $K(h, h)=S(h, h)\neq 0$ and if

$$\int_0^h |S_x'(h, x)/S(x, x)|dx<N,$$

$$\int_0^h |K_x'(h, x)/K(x, x)|dx\leqq q<1,$$

where N and q are independent of h, then $(*)$ implies $(*')$. The special case $K(h, x)=2h/(x^2+h^2)$, $S(h, x)=1/h$ was proposed as a problem by Natanson. *A. Zygmund.*

Polynomials, Polynomial Approximations

Marden, Morris. A refinement of Pellet's theorem. Bull. Amer. Math. Soc. **54**, 550–557 (1948).

The author proves the following refinement of a theorem due to A. Pellet [Bull. Sci. Math. (2) **5**, 393–395 (1881)]. If the polynomial $f(z)=a_0+a_1z+\cdots+a_pz^p+\cdots+a_nz^n$, $a_p\neq 0$, is such that the real polynomial $F_p(z)=|a_0|+|a_1|z+\cdots+|a_{p-1}|z^{p-1}-|a_p|z^p+|a_{p+1}|z^{p+1}+\cdots+|a_n|z^n$ has two positive zeros r and R with $r<R$, then the polynomial $\Phi_p(z)=|a_1|+|a_2|z+\cdots+|a_{p-1}|z^{p-2}-|a_p|z^{p-1}+|a_{p+1}|z^p+\cdots+|a_n|z^{n-1}$ has also two positive zeros r_0 and R_0 with $r_0<r<R<R_0$. Furthermore, $f(z)$ has exactly p zeros in or on the curve $G(r_0, r; p, \alpha_0)$, where $\alpha_0=\arg a_0/a_p$, and no zeros between the curves $G(r_0, r; p, \alpha_0)$ and $G(R, R_0; p, \alpha_0+\pi)$. The curves G bound gear-wheel regions described by S. Lipka [Monatsh. Math. Phys. **50**, 125–127 (1941); these Rev. **5**, 225]. The theorem is applied to the refinement of various known bounds on the zeros of a polynomial. The theorem is also generalized by the replacement of the polynomial $\Phi_p(z)$ by the polynomial $F_p(z)-|a_k|z^k$, $k\neq p$, and by the replacement of the polynomial $f(z)$ by a power series.
E. Frank (Chicago, Ill.).

Colombo, G. Intorno alla distribuzione degli zeri di certi polinomi. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **3**, 530–535 (1947).

This paper concerns the roots of the polynomial $P(x)+iQ(x)=R(x)$, where P, Q are polynomials of the same degree with initials p, q; the roots of P, Q are on an oval C in the plane of the complex variable; the roots of P separate those of Q on C. It is shown that the roots of R lie within or on a curve which is the locus of the intersection of supports of C meeting at angle Θ, where $0\leqq\Theta\leqq\pi$ and $\pm\Theta$ is one of the values of the amplitude of q/pi. A corollary is that when the coefficients and roots of P, Q are real the roots of R lie within or on the boundary of a semicircle whose diameter is an interval of the real axis containing all the roots of P, Q; if the least roots of P, Q are a, b, the semicircle lies in the half-plane having the sign of $(b-a)pq$.
J. M. Thomas (Durham, N. C.).

Grossmann, K. H. Elementare Begründung und Verschärfung des Hurwitzschen Stabilitätskriteriums. Schweiz. Arch. Angew. Wiss. Tech. **14**, 242–247 (1948).

In this note, a system is defined to have the quality q $(0<q)$ of stability if all the zeros of the corresponding characteristic polynomial $f(z)$ lie in the half-plane $\Re(z)<-q$. For stability of a given quality q, it is necessary and sufficient that the polynomial $g(Z)=f(Z-q)$ have all its zeros in the half-plane $\Re(Z)<0$. Hence various criteria for a stability of quality q may be derived at once from the well-known criteria of Routh, Hurwitz and Nyquist for stability in the usual sense $q=0$. *M. Marden* (Milwaukee, Wis.).

Leja, François. Sur une propriété des polynomes de Lagrange. C. R. Acad. Sci. Paris **226**, 1416–1417 (1948).

Suppose F is a curve situated on the frontier of a domain D which contains the point at infinity and that on F, corresponding to each positive integer n, there is a system of $n+1$ points $\xi^{(n)}=\{\xi_0^{(n)}, \xi_1^{(n)}, \cdots, \xi_n^{(n)}\}$. Further suppose that the curve F is divided into two curves F_1 and F_2 without points in common and that the first $\mu=\mu(n)$ of the points $\{\xi^{(n)}\}$ lie on F_1 and the remainder on F_2. Let

$$L_n^{(i)}(z; \xi^{(n)})=\prod_{k=0, k\neq i}^{n}(z-\xi_k^{(n)})/(\xi_i^{(n)}-\xi_k^{(n)})$$

be the Lagrangian polynomials corresponding to the system $\{\xi^{(n)}\}$ and write $M_n^{(1)}(z)=\max|L_n^{(i)}(z; \xi^{(n)})|$ for $i=0, 1, \cdots, \mu-1$, and $M_n^{(2)}(z)=\max|L_n^{(i)}(z; \xi^{(n)})|$ for $i=\mu, \cdots, n$; then the author proves the following theorem. If z_0 is a point of the curve F_2 and if $\limsup_{n\to\infty}[M_n^{(2)}(z_0)]^{1/n}<1$ then $\limsup_{n\to\infty}[M_n^{(1)}(z_0)]^{1/n}>1$. This theorem depends on an earlier result of the author [Ann. Soc. Polon. Math. **12**, 57–71 (1934); **18**, 4–11 (1945); these Rev. **8**, 255].
A. C. Offord (London).

Gončarov, V. L. On an interpolation scheme of general form. Doklady Akad. Nauk SSSR (N.S.) **59**, 1529–1532 (1948). (Russian)

Let $U[F]=\int F(x)d\psi(x)$ denote an integral taken over a point set E in the complex plane corresponding to a distribution of complex mass determined by $\psi(x)$. The set E may be either a plane set or a set situated on a curve. Suppose we have a set of $n+1$ such integrals $U_m[F]=\int F(x)d\psi_m(x)$ $(m=0, 1, \cdots, n)$ subject only to the condition that there is no polynomial of degree n (other than $P_n(x)\equiv 0$) for which all the equations $U_m[P_n]=0$ $(m=0, 1, \cdots, n)$ are satisfied identically. The author discusses the following problem: to find a polynomial $P_n(f, x)=P_n(x)$ of degree n such that $U_m[P_n]=U_m(f)$ $(m=0, 1, \cdots, n)$ and to estimate the difference $R_n(f, x)=f(x)-P_n(f, x)$. He gives formulae for both $P_n(x)$ and $R_n(x)$. *A. C. Offord* (London).

Special Functions

Jaeger, J. C. Repeated integrals of Bessel functions and the theory of transients in filter circuits. J. Math. Physics **27**, 210–219 (1948).

Solutions of certain problems in electric circuit theory involve the repeated integral of Bessel functions

$$Ji_{n,r}(t)=\int_0^t dt_r\int_0^{t_r}dt_{r-1}\cdots\int_0^{t_2}dt_1J_n(t_1)$$

and the "highpass-filter function"

$$\Phi_n(t)=\int_0^\infty J_0[2(ut)^{\frac{1}{2}}]J_n(u)du.$$

By means of the technique of the Laplace transform the author investigates a number of transient problems. He finds that for the simplest ten semi-infinite ladder networks the solutions can be expressed as single integrals involving tabulated functions, in view of the tables presented here, of $2^{-r}Ji_{0,r}(t)$ to 8 decimals, and of $\Phi_n(t)$ and $d\Phi_n/dt$ to 5 decimals [$t=0(1)24$; $r=1(1)7$; $n=0(1)7$]. *C. J. Bouwkamp.*

Eberlein, W. F. Characteristic values of spheroidal wave functions. Physical Rev. (2) **74**, 190–191 (1948).

The author considers the equation of prolate spheroidal wave functions ($c>0$), $(1-x^2)y''-2(m+1)xy'+(b-c^2x^2)y=0$, in which he substitutes $y(z)=\sum_0^\infty a_j\psi_j(z)$, $z=c^{\frac{1}{2}}x$, where $\psi_j(z)=(-1)^j\exp(z^2/2)(d/dz)^j\exp(-z^2)$, so as to obtain a five-term recursion relation for the a's. Setting $b=(2l+1)c+\sum_0^\infty\beta_sc^{-s}$, $a_l=1$, $a_j=\sum_1^\infty\alpha_sc^{-s}$, $(j\neq l)$ and equating to zero the various powers of c, the author obtains the α's and β's. The first few coefficients β are given explicitly. The result was obtained independently by J. Meixner [Z. Angew. Math. Mech. **25/27**, 137–138 (1947); these Rev. **9**, 431]. A critical discussion of the procedure and the character of the resulting series is postponed to a later paper.

C. J. Bouwkamp (Eindhoven).

Titchmarsh, E. C. Some integrals involving Hermite polynomials. J. London Math. Soc. **23**, 15–16 (1948).

The author gives the formula

$$\int_{-\infty}^{\infty}e^{-2x^2}H_m(x)H_n(x)H_p(x)dx=\pi^{-\frac{1}{2}}2^{s-1}\Gamma(s-m)\Gamma(s-n)\Gamma(s-p)$$

where m, n, p are nonnegative integers and $m+n+p=2s-1$ is even. The integral vanishes if $m+n+p$ is odd.

A. Erdélyi (Edinburgh).

Shanker, Hari. Some definite integrals involving confluent hypergeometric functions. J. London Math. Soc. **23**, 44–49 (1948).

By means of the product theorem of operational calculus the author evaluates various integrals with Bessel functions, parabolic cylinder functions, and confluent hyperbolic functions. A typical example is

$$2(ab)^{\frac{1}{2}}\{\Gamma(2m+1)\}^2\int_0^{\frac{1}{2}\pi}e^{\frac{1}{2}(a+b)\cos 2\theta}J_{2m}\{(ab)^{\frac{1}{2}}\sin 2\theta\}\tan^{2k}\theta d\theta$$
$$=\Gamma(\tfrac{1}{2}+k+m)\Gamma(\tfrac{1}{2}-k+m)M_{k,m}(a)M_{k,m}(b).$$

A. Erdélyi (Edinburgh).

Erdélyi, A. Transformations of hypergeometric functions of two variables. Proc. Roy. Soc. Edinburgh. Sect. A. **62**, 378–385 (1948).

The set of linear transformations of complete hypergeometric functions of two variables of order two are shown to be derivable from transformations of contour integral representations. It is shown that with three possible exceptions the set of functions may be expressed in terms of Appell's series F_2. Certain quadratic transformations are developed in a similar manner, and the application of the method to generalized hypergeometric functions of many variables is outlined. *N. A. Hall* (Minneapolis, Minn.).

Wintner, Aurel. The real elliptic ϑ_3-function. Duke Math. J. **15**, 407–411 (1948).

The author proves that for every q with $0<q<1$ the graph of the even periodic function

$$y=\theta_q(x)=1+2\sum_{n=1}^{\infty}q^{n^2}\cos nx$$
$$=2\pi^{\frac{1}{2}}p^{-1}\sum_{n=-\infty}^{\infty}\exp[-(x+2\pi n)^2p^{-2}],\qquad p=(-4\log q)^{\frac{1}{2}},$$

has just one pair of inflection points over a period. More specifically, d^2y/dx^2 is negative or positive according as $0\leqq|x|<x_q$ or $x_q<|x|\leqq\pi$, where x_q is a unique point of the interval $0<x<\pi$ whenever $0<q<1$. First it is shown by a majoration process that in

$$\tfrac{1}{2}\pi^{-\frac{1}{2}}p^3\theta_q''(x)=(4x^2/p^2-2)\exp(-x^2/p^2)+\varphi_p(x)$$

the first term on the right hand side preponderates in absolute value over $\varphi_p(x)$. From Rouché's theorem and the general theories of Sturm or Laguerre it follows that $\theta_q''(x)$ cannot have more than two zeros on the interval $-\pi<x\leqq\pi$, $0<q<1$. The main result is proved by the observation that the graph of the periodic $y=\theta_q(x)$, being symmetric with respect to the y-axis, cannot have less than two inflection points. Furthermore, it follows that $\theta_q(x)$ is decreasing if $0<x<\pi$. [Remark: in line 10 from the bottom on p. 408 read $\exp(-4\pi nzp^{-2}-4\pi^2n^2p^{-2})$ instead of $\exp(-4\pi zp^{-2}+4\pi^2n^2p^{-2})$.] *S. C. van Veen* (Delft).

Hartman, Philip. The zeros of the derivatives of the real elliptic ϑ_3-function. Duke Math. J. **15**, 413–420 (1948).

The object of this paper is to investigate the zeros of the higher derivatives of the function $\theta_q(x)$ of the preceding review. If the number of zeros of $\theta_q^m(x)$ in a period $-\pi<x\leqq\pi$ (counting multiplicities) is $N_m(q)$, it follows by Rolle's theorem that $N_{m+1}(q)\geqq N_m(q)$ $(0<q<1)$. It follows from the general theories of either Sturm or Laguerre that $N_m(q)$ is a nondecreasing step-function of q on $0<q<1$. It is shown that, if $m>0$, (1) $N_{2m-1}(q)=N_{2m}(q)$ $(0<q<1)$; (2) $N_{2m}(q)$ has exactly m values, namely, $2, 4, \cdots, 2m$; (3) $N_{2m}(q)$ is continuous from the right (and, if $m>1$, its discontinuities separate and are separated by those of $N_{2m-2}(q)$; (4) the power series $P_m(q)=\sum_{n=1}^{\infty}(-1)^nn^{2m}q^{n^2}$ has exactly m zeros $0=q_0<q_1<\cdots<q_{m-1}$ on $-1<q<1$; each of these zeros is simple; (5) the $m-1$ positive zeros of $P_m(x)$ are the discontinuity points of $N_{2m}(q)$; (6) the zeros of $\theta_q^{2m}(x)$ on $|x|<\pi$ are simple, and $x=\pi$ is a double zero or is not a zero according as q is or is not a zero of $P_m(q)$; (7) the zeros of $\theta_q^{2m+1}(x)$ on $|x|<\pi$ are simple, and $x=\pi$ is a triple or a simple zero according as q is or is not a zero of $P_{m+1}(q)$. *S. C. van Veen* (Delft).

Harmonic Functions, Potential Theory

Aronszajn, N. Recherches sur les fonctions harmoniques dans un carré. J. Math. Pures Appl. (9) **27**, 87–175 (1948).

Let $h(z)$ be a function in the rectangle (R) $-a<x<a$, $-b<y<b$ $(z=x+iy)$. The central problem of the paper is that of the possibility of a decomposition $h(z)=h_1(z)+h_2(z)$, $z\in R$, where $h_1(z)$ would be harmonic in the strip $-b<y<b$ and of period $2a$ in x, and $h_2(z)$ would be harmonic in the strip $-a<x<a$ and of period $2bi$ in y. It is shown that such a decomposition is always possible, and that the result holds if the word "harmonic" is replaced throughout by "analytic." Next it is shown that, if $h(z)$ is of the class L^2 in R, there exist functions h_1, h_2 also of the class L^2, provided $h(z)$ is either even-even or odd-odd, but not necessarily if, e.g., $h(z)$ is even-odd. The corresponding problem for other classes L^p is open, but the problem of uniqueness is solved completely. Connections are established between harmonic functions of the class L^2 in R and the Fourier series $\sum(a_n\cos nx+b_n\sin nx)$ such that $\sum n(a_n^2+b_n^2)<\infty$. Extensions are given to functions harmonic in space.

A. Zygmund (Chicago, Ill.).

Verblunsky, S. Inequalities for the derivatives of a bounded harmonic function. Proc. Cambridge Philos. Soc. **44**, 155–158 (1948).

The author suggests the use of the Poisson integral together with a lemma on the rearrangement of a function to prove inequalities for bounded harmonic functions in the unit circle. As applications, he proves that, for such a function $h(r,\theta)$, with $|h|\leqq 1$, $h(0)=c$,

$$\partial h/\partial r\leqq\frac{4\pi^{-1}\cos\frac{1}{2}\pi c}{1+r^2+2r\sin\frac{1}{2}\pi c},$$

with a more complicated result for $\partial h/\partial\theta$. The extremal functions are found. *R. P. Boas, Jr.* (Providence, R. I.).

Picone, Mauro. Sur la théorie d'une équation aux dérivées partielles classique de la physique mathématique. C. R. Acad. Sci. Paris **226**, 1945–1947 (1948).

Let A be a domain in Euclidean space E_n, $n\geqq 3$, and let Γ be an open subarc of a regular curve, such that Γ is part of the frontier $\mathfrak{F}A$ of A. Bouligand [Bull. Sci. Math. (2) **47**, 386–397 (1923)] and Lebesgue [same C. R. **176**, 1097–1099, 1270–1271 (1923)] have shown that, if Γ is an isolated part of $\mathfrak{F}A$, then a function $h(x_1,\cdots,x_n)$ harmonic and bounded in A can be extended so as to be harmonic in $A+\Gamma$. The author extends this result; he replaces h by functions u satisfying the equation $\sum_{i=1}^{n}\partial^2u/\partial x_i^2+gu=0$, where g is defined in A, and Γ by a regular q-variety without boundary, $1\leqq q\leqq n-2$. *M. O. Reade* (Ann Arbor, Mich.).

***Weinstein, Alexander. On generalized potential theory and on the torsion of shafts.** Studies and Essays Presented to R. Courant on his 60th Birthday, January 8, 1948, pp. 451–460. Interscience Publishers, Inc., New York, 1948. $5.50.

The author considers fundamental solutions φ and their "associates" ψ satisfying systems of equations of the form $\tau(x,y)\varphi_x=\psi_y$, $\tau(x,y)\varphi_y=-\psi_x$. He points out in particular that the associated functions ψ are in general multivalued [Weinstein, Bull. Amer. Math. Soc. **52**, 240, 431–432 (1946)], a fact which, he states, was not recognized by Beltrami in his collected works [vol. III] and thus led to many erroneous results. Some applications to fluid dynamic problems are made. It should be pointed out that reference to the multivaluedness of the associated functions ψ is also made by Bergman [Duke Math. J. **14**, 349–366 (1947); Trans. Amer. Math. Soc. **62**, 452–498 (1947); these Rev. **9**, 181; **10**, 162]. *A. Gelbart* (Syracuse, N. Y.).

Weinstein, Alexander. On axially symmetric flows. Quart. Appl. Math. **5**, 429–444 (1948).

In this paper, which is a further development of the author's paper reviewed above, the author gives other examples of the use of the fundamental solutions φ and their associated functions ψ. He investigates, in particular, the stream functions for the flows around a homogeneous circumference and a disc with different distributions of sources and sinks. The potentials here involved play a similar role to the potential of a point source in the general case of an arbitrary three-dimensional flow. [For some earlier works on the general theory of multivalued three-dimensional harmonic functions see Bergman, Math. Ann. **99**, 629–659 (1928); **101**, 534–558 (1929); Bull. Amer. Math. Soc. **49**, 163–174 (1943); Marden, Amer. J. Math. **67**, 109–122 (1945); these Rev. **4**, 159; **7**, 225.] *A. Gelbart.*

Brelot, Marcel. Quelques propriétés et applications du balayage. C. R. Acad. Sci. Paris **227**, 19–21 (1948).

The author establishes additional properties of the "exterior sweeping-out" process introduced by H. Cartan [see Ann. Univ. Grenoble. Sect. Sci. Math. Phys. (N.S.) **22**, 221–280 (1946); these Rev. **8**, 581]; the following is typical. Let Ω be a domain in n-space $(n\geqq 2)$, with nonpolar complement. Then the "sweeping-out" of a unit mass placed at a point P of set B, $B\subset\Omega$, does not transport any mass onto the "fine" interior of B, except for P. The author makes various applications of his new results, to obtain results similar to the following. If u is a G-potential (due to masses of variable sign), then u^+, u^- and $|u|$ are G-potentials (due to masses of constant sign). The author uses the "fine" topology of H. Cartan, which is the least "fine" topology on Ω which makes all functions subharmonic on Ω continuous on Ω. *M. O. Reade* (Ann Arbor, Mich.).

Kamke, E., und Lorentz, G. G. Über das Dirichletsche Prinzip. Math. Z. **51**, 217–232 (1948).

Let G be a bounded domain, in n-space, with boundary R. Let x denote a point of the space. The object of the paper is to discuss the relationship between the following two problems. (i) Boundary value problem: find a function which is harmonic in $G+R$ and takes on assigned values $f(x)$ on R. (ii) Variational problem: among all admissible functions, $v(x)$ with assigned boundary values $f(x)$, find one whose Dirichlet integral is a minimum. A function $v(x)$ is admissible if it satisfies the following conditions: (a) $v(x)$ is continuous in $G+R$; (b) the partial derivatives of $v(x)$ exist in G, except perhaps on a subset of finite $(n-1)$-dimensional Carathéodory measure; (c) the Dirichlet integral of $v(x)$ exists. The main result of the paper is that if there exist admissible functions $v(x)$ at all with the assigned boundary values $f(x)$, then the two problems are equivalent. This result represents a substantial improvement over previous literature. *T. Radó* (Columbus, Ohio).

Consiglio, A. Su la risoluzione in termini finiti dei problemi di Schwarz e di Dirichlet relativi al cerchio. Matematiche, Catania **2**, 1–8 (1946).

La fonction holomorphe dans le cercle $|z|<a$, dont la partie réelle prend les valeurs $\varphi(\theta)$ sur la frontière γ, peut s'exprimer (à une constante imaginaire près)

$$(2\pi i)^{-1}\int_\gamma \varphi(\theta)\{2(\zeta-z)^{-1}-\zeta^{-1}\}d\zeta_\theta.$$

L'auteur exprime $\cos\theta$ et $\sin\theta$ au moyen de l'affixe ζ du point θ de γ et déduit, d'un développement de φ en série de Fourier $\sum(a_n\cos n\theta+b_n\sin n\theta)$, l'expression de φ: $\frac{1}{2}\sum(a_n-ib_n)(\zeta/a)^n+(a_n+ib_n)(a/\zeta)^n$. Pour obtenir une substitution intéressante dans l'intégrale, il considère ce $\sum$ pour ζ dans le cercle mais sans examiner les conditions sous lesquelles ce $\sum$ peut avoir alors un sens. *M. Brelot.*

Greenberg, H. J. The determination of upper and lower bounds for the solution of the Dirichlet problem. J. Math. Physics **27**, 161–182 (1948).

Let C be a simple closed contour in the (x, y)-plane, consisting of a finite number of regular arcs, and let D denote the domain interior to C. Let s measure arc length along C and let n be the inwardly directed normal to C. The author uses two variational principles for the determination of upper and lower bounds of the value $w(x_0, y_0)$ of a solution $w(x, y)$ of the Dirichlet problem $\Delta w=0$ in D, $w=f$ on C: the Dirichlet principle and a complementary principle that was obtained by K. O. Friedrichs [Nachr. Ges. Wiss. Göttingen. Math.-Phys. Kl. **1929**, 13–20]. The latter is as follows. Let V be the class of all functions v satisfying Laplace's equation $\Delta v=0$ in D. Then for any $v\in V$,

$$D^*(w)\leqq D^*(v),\quad D^*(v)=\tfrac{1}{2}\int_D(v_x^2+v_y^2)dA+\int_C f(\partial v/\partial n)ds.$$

These two variational principles are applied to the boundary value problem $\Delta\omega=0$ in D, $\omega=f-P\log(1/r)$ on C, where P is a real constant which plays the role of a parameter.

For obtaining bounds for $w(x_0, y_0)$ and for the elimination of integrals containing $\partial w/\partial n$ or $\partial\omega/\partial n$ the author uses the representation theorem for a harmonic function in terms of its boundary values and those of its normal derivatives. The upper and lower bounds obtained in this way for $P\cdot w(x_0, y_0)$ are polynomials quadratic in P. Using the inequality characterizing a definite quadratic form, the author gets upper and lower bounds without P. By an iteration procedure the bounds can be improved. With slight changes one can adapt the method to find bounds for the analogous boundary value problem associated with the Poisson equation. The results are applied to the problem $\Delta w=-1$ in D, $w=0$ on C, where D is a square and (x_0, y_0) is its center. *P. Funk.*

Zahorski, Zygmunt. On a problem of M. F. Leja. Ann. Soc. Polon. Math. **20** (1947), 215–222 (1948).

Considérons dans le plan un ensemble borné fermé F. Leja dit que F possède en $z_0\in F$ la propriété V si, à toute suite de polynomes $P_n(z)$ (de degré n) uniformément bornés sur F, et à tout $\epsilon>0$, on peut associer un voisinage de z_0 sur lequel $P_n(z)(1+\epsilon)^{-n}$ est uniformément borné. L'auteur montre par un exemple que, même si F est de capacité positive en chacun de ses points, il n'y a pas toujours la propriété V. Cela résulte d'ailleurs du critère (nécessaire et suffisant) donné ultérieurement par Leja [voir l'analyse suivante]. *M. Brelot* (Grenoble).

Leja, F. Une condition de régularité et d'irrégularité des points frontières dans le problème de Dirichlet. Ann. Soc. Polon. Math. **20** (1947), 223–228 (1948).

Reportons nous à la définition donnée dans l'analyse qui précède. L'auteur montre que si z_0 est un point de la frontière bornée F d'un domaine plan D contenant le point à l'infini, la propriété V est caractérisée par la régularité du point z_0 (au sens du problème de Dirichlet pour D). La propriété V entraîne la régularité grâce à des travaux antérieurs de l'auteur sur le diamètre transfini et la fonction de Green [mêmes Ann. **12**, 57–71 (1934); **18**, 4–11 (1945); ces Rev. **8**, 255]. La réciproque, qui est établie en utilisant aussi divers résultats antérieurs de l'auteur, me paraît dériver plus simplement de la remarque suivante: la fonction sousharmonique à distance finie $\log|P_n(z)|$ est majorée dans D à une constante près indépendante de n par n fois la fonction de Green de D et de pôle à l'infini. L'auteur propose en terminant une extension à l'espace où $\log|P_n|$ serait remplacé par $\Phi_n=C_n-\sum_{k=1}^{n}1/(PP_k^{(n)})$ (C_n constante, $P_k^{(n)}$ point fixe, P point courant de Φ_n); Φ_n est encore supposé borné supérieurement uniformément sur F et il chercherait à ce que $\Psi_n=\Phi_n-n\epsilon$ fût aussi borné supérieurement uniformément dans un voisinage de $P_{0\epsilon}F$, $\epsilon>0$ étant fixé. Mais le raisonnement que je viens d'indiquer pour la réciproque est facile à adapter à condition de prendre $\Psi_n=\Phi_n-C_n\epsilon$ (le $\sum$ du Φ_n pouvant être un potentiel quelconque de masses positives). *M. Brelot* (Grenoble).

Dubošin, G. N. Expansion of the potential of the elliptical Gaussian ring. Vestnik Moskov. Univ. no. **10**, 59–67 (1947). (Russian)

The author considers the potential of a heavy ellipse $R=p/(1+e\cos w)$. The density at any point is proportional to the square of the radius vector R. The potential at a point with cylindrical coordinates (z, ρ, v) is expanded in a double power series in z and ρ. The cases $\rho<R$ and $\rho>R$ give different series. The series are valid only for $|z|^2<\min\{\rho^2+R^2-2\rho R\cos(v-w)\}$. *W. H. Muller.*

Bolder, H. Une extension de la loi de réciprocité pour l'énergie dans la théorie du potentiel et quelques applications. II. Nederl. Akad. Wetensch., Proc. **51**, 548–555 = Indagationes Math. **10**, 193–200 (1948).

[For part I cf. same Proc. **51**, 496–502 = Indagationes Math. **10**, 180–186 (1948); these Rev. **9**, 508.] The author proposes as an application of his generalized law of reciprocity the problem of determining in terms of its potential the moment of a double distribution on the x-axis with elements oriented normally to the x-axis. He also comments briefly on the corresponding problem for a circle. In each case he notes that the formulas obtained are those previously given by A. Monna [same Proc. **50**, 1056–1062 = Indagationes Math. **9**, 470–476 (1947); these Rev. **9**, 352]. *F. W. Perkins* (Hanover, N. H.).

***Soudan, Robert. Étude de la Déformation d'un Corps à Potentiel Constant.** Thesis, University of Geneva, 1940. 43 pp.

The material in this thesis is related to that in two other papers [R. Soudan, C. R. Soc. Phys. Genève **57**, 79–82 (1940); K. Bleuler, ibid. **58**, 220–221 (1941); these Rev. **4**, 44; **3**, 244]. The author formulates the following two problems: (1) Given a body of constant density δ, is it

possible to deform the body continuously so as to preserve homogeneity, total mass, and potential at exterior points? (2) Given a surface S bounding a body of constant density δ and potential U, and a second surface S_1, exterior to S. Replace δ by another constant δ_1 and fill the space between S and S_1 with a distribution of variable density δ_2 yielding a combined potential U_1. Is it possible to choose δ_1 and δ_2 so that (at exterior points) we have $U=U_1$?

These problems are not solved in this general form, but related results are obtained. In this limited treatment attention is mainly confined to the corresponding problems for the logarithmic potential. The boundary Γ (corresponding to S) is assumed analytic and regular, and in some parts of the theory is further restricted. The author investigates infinitesimal deformations with potential stationary of the nth order. It is found there exist deformations in which an arbitrary point $P:(\rho, \theta)$ of the boundary is displaced by the amount $\epsilon i(\theta, \epsilon)$, where ϵ is an infinitesimal and $i(\theta, \epsilon)$ is bounded, such that the potential due to a fixed mass distributed uniformly over the variable lamina is altered by an amount not greater than $A(n)\epsilon^n$ at exterior points at a distance at least $k>0$ from Γ. The author observes that to establish the existence of a fixed potential by his method we need further information concerning the convergence of a certain series. The author also studies, for the logarithmic potential, a problem of a type similar to (2). Results corresponding to those outlined above are obtained.

F. W. Perkins (Hanover, N. H.).

***Bilger, Gérard. Sur les Polygones et les Polyèdres de Même Potentiel.** Thesis, University of Geneva, 1942. 39 pp.

In the introduction the author discusses properties of Wavre's "functions of passage" which play an important role in the later developments [R. Wavre, Vierteljschr. Naturforsch. Ges. Zürich **85** Beiblatt (Festschrift Rudolf Fueter), 87–94 (1940); these Rev. **2**, 293]. These functions are concerned with the difference between the analytic continuation of a Newtonian (or logarithmic) potential function across a singular surface (or curve) and the actual value of the potential function; they are useful in establishing the equivalence of potentials of which the explicit computation in finite form is impractical.

The logarithmic potential of distributions on polygons is studied. For example, one theorem states that the potential due to a uniform distribution of mass on the sides of a regular polygon yields the same potential as an equal mass distributed uniformly on the rays joining the center to the vertices. In some of the theorems it is not required that the polygons be convex. Curvilinear polygons are also considered. A number of results are given for logarithmic potentials due to uniform distributions of mass on the areas bounded by polygons, and the Newtonian potentials due to homogeneous polyhedra. Also a number of geometric properties of polygons and polyhedra are discussed as applications.

The author notes that the subject of this thesis is related to that of the thesis of R. Soudan [see the preceding review]. Related material is also found in a paper published after the present thesis [G. Choquet and J. Deny, Bull. Soc. Math. France **72**, 118–140 (1944); these Rev. **7**, 161].

F. W. Perkins (Hanover, N. H.).

Bullard, E. C., and Cooper, R. I. B. The determination of the masses necessary to produce a given gravitational field. Proc. Roy. Soc. London. Ser. A. **194**, 332–347 (1948).

The problem studied in this paper is indeterminate and the two methods described in it give the solution under two assumptions: the observed gravitational anomaly (field) is ascribed to a special distribution of disturbing masses in the form of a very thin horizontal layer of variable surface density which is considered as the unknown of the problem and, moreover, this mass distribution is located at a specified and known depth. A family of solutions for different depths can thus be obtained.

In the first method the observed field is extrapolated downwards by a finite difference procedure. In the second a solution is obtained in the form of an integral which can be evaluated numerically with the aid of tables of functions. Examples of application for both methods are given and discussed. *E. Kogbetliantz* (New York, N. Y.).

Differential Equations

Nordon, Jean. Sur une méthode de calcul des images symboliques. C. R. Acad. Sci. Paris **227**, 23–25 (1948).

A few remarks on the solution of ordinary linear differential equations with variable coefficients by operational methods are illustrated by a discussion of various differential equations whose solutions can be expressed in terms of Bessel functions. The only new feature is the [unmotivated] introduction of "finite parts" of divergent integrals.

A. Erdélyi (Edinburgh).

A letter of N. G. Čebotarev on a mathematical problem arising in connection with the estimated deviation of the coordinates of a regulator if the perturbing force is restricted in modulus. Avtomatika i Telemehanika **9**, 331–334 (1948). (Russian)

It is shown that the solution of

$$x^{(n)}+a_1x^{(n-1)}+\cdots+a_nx=f(t),$$

subject to $x(0)=0$, $x'(0)=0$, $\cdots$, $x^{(n-1)}(0)=0$, satisfies the inequality $|x|\leqq c$ provided that the roots of the characteristic equation have negative real parts and $|f(t)|\leqq 1$. The proof depends upon the representation of the solution of the inhomogeneous equation in terms of the solution of the homogeneous equation. *R. Bellman.*

Sarantopoulos, Spyr. A generalization of Legendre's equation and of Laplace's integral. Bull. Soc. Math. Grèce **23**, 30–50 (1948). (English. Greek summary)

The differential equation satisfied by the Euler transformation

$$u=(2\pi i)^{-1}\int_C e^{\phi(t)}[f(t)]^n(t-z)^{-n-1}dt$$

is found to be

$$f(z)u^{(\mu)}(z)-\sum_{\nu=1}^{\mu}[1/(\nu-1)!]\times\prod_{j=1}^{\nu-1}(n+\mu-j)[f^{(\nu)}(z)+\psi^{(\nu-1)}(z)]u^{(\mu-\nu)}(z)=0,$$

where $f(z)$ is a polynomial of order 2^{n+1} and ψ is related to f:

$$f(z)=a_0z^{n+1}+a_1z^n+\cdots+a_nz+a_{n+1},$$
$$\psi(z)=f\phi'=-(\mu n+1)a_0z^n+a_1z^{n-1}+\cdots+a_{n-1}z+a_n,$$

so that ϕ is determined in terms of f. The properties of the possible u's are discussed in some detail for various cases.

H. Feshbach (Cambridge, Mass.).

Fel'dbaum, A. A. On the distribution of the roots of the characteristic equations of systems of regulation. Avtomatika i Telemehanika 9, 253–279 (1948). (Russian)

The author considers the linear differential systems associated with several electronic circuits and illustrates the influence of the location of the roots of the characteristic equation upon the behavior of the circuit. *R. Bellman.*

Ansoff, H. I., and Krumhansl, J. A. A general stability criterion for linear oscillating systems with constant time lag. Quart. Appl. Math. 6, 337–341 (1948).

The authors study the stability of a feedback system which is governed by an equation of the form

$$Iy''(t)+Ry'(t)+Ky(t)=-Sy^{(n)}(t-\tau),$$

where I, R, K, S and τ are positive constants. The system is stable if, and only if, the equation $Ip^2+Rp+K+Sp^ne^{-\tau p}=0$ has no roots in the right-hand half of the p-plane. The well-known Nyquist criterion (essentially Cauchy's index theorem) is used to determine whether or not there are any roots with positive real parts. The cases in which $n=0, 1, 2$ are investigated in considerable detail; and it is shown that, if $n>2$, the system is necessarily unstable.

L. A. MacColl (New York, N. Y.).

Foà, Enzo. Sull'equazione del moto smorzato con parametri variabili e su un caso di instabilità. Pont. Acad. Sci. Acta 6, 9–15 (1942).

In the first part of this paper the author considers differential equations of the form $y''+2p(t)y'+[q(t)]^2y=0$, where the functions p and q satisfy the relations $p>0$, $p^2-q^2<0$. He describes a familiar method for obtaining approximate solutions, which is based on the transformation of variables

$$y=x\exp\left[-\int_0^t p\,dt\right],\quad x=(2J/\omega)^{\frac{1}{2}}\sin\theta,$$

$$dx/dt=(2J\omega)^{\frac{1}{2}}\cos\theta,\quad \omega^2=q^2-p^2-dp/dt,$$

and he gives estimates for the magnitude of the error.

The second part of the paper is devoted to a study of equations of the form $y''+2py'+(A^2e^{2\lambda t}+p^2)y=0$, where A, p, λ are positive constants. It is shown that, if $2p>\lambda$, both y and dy/dt approach zero as t approaches infinity. On the other hand, if $2p<\lambda$, as t approaches infinity y approaches zero, and dy/dt attains values which are arbitrarily large. *L. A. MacColl* (New York, N. Y.).

Manacorda, Tristano. Sul comportamento asintotico degli integrali della equazione: $y''(x)+\alpha(x)y'(x)+\beta(x)y(x)=0$. Ann. Mat. Pura Appl. (4) 26, 73–83 (1947).

The author studies the asymptotic behavior of the solutions of the equation of the title under the hypothesis that $\beta(x)\to 0$ as $x\to+\infty$. His results are too detailed to state simply. *R. Bellman* (Stanford University, Calif.).

Ascoli, Guido. Sopra un caso di stabilità per l'equazione $y''+A(x)y=0$. Ann. Mat. Pura Appl. (4) 26, 199–206 (1947).

The author derives asymptotic forms for the solutions of the equation of the title. Most of his results are superseded by results of Hartman, of Wintner, and of Levinson [Duke Math. J. 15, 111–126 (1948); these Rev. 9, 509].

R. Bellman (Stanford University, Calif.).

Biernacki, M. Sur un problème d'interpolation relatif aux équations différentielles linéaires. Ann. Soc. Polon. Math. 20 (1947), 169–214 (1948).

Given a linear equation (1) $y^{(n)}+a_{n-1}y^{(n-1)}+\cdots+a_0y=0$, with real constant coefficients, it may or may not be true that for every choice of k real points $x_1, \cdots, x_k$ $(1\leqq k\leqq n)$, and with an arbitrary assignment of real values at these points, there exists a solution of (1) which at these points takes on these values. If the equation has this property for a specific k it is said to be of the type L_k; in the contrary case it is of the type S_k. Every equation is of the type L_1. This paper is devoted to the following question. Under what conditions upon the roots of the characteristic equation (2) $S^n+a_{n-1}S^{n-1}+\cdots+a_0=0$, is (1) of the type L_k or S_k for various k? The following are examples of theorems that are derived. (i) A necessary and sufficient condition that (1) be of the type S_2 is that all roots of (2) be simple, that they all have the same real part, and that the ratios of their imaginary parts be all rational. (ii) A necessary and sufficient condition that (1) be of the type L_n is that the roots of (2) be all real. (iii) The equation (1) is of the type S_{n-1} if (2) has at least two pairs of complex roots the ratio of whose imaginary parts are rational. Many similar results are given. *R. E. Langer* (Madison, Wis.).

Quade, W. Schranken für die Eigenwerte der Sturmschen Randwertaufgabe zweiter Ordnung. Math. Z. 51, 92–125 (1948).

The Sturmian boundary problem

$$(1)\qquad \begin{cases} y''+Q(x,\lambda)y=0, & \\ \alpha y(a)+\alpha' y'(a)=0, & \alpha^2+\alpha'^2>0, \\ \beta y(b)+\beta' y'(b)=0, & \beta^2+\beta'^2>0, \end{cases}$$

is cast into the form

$$(2)\qquad \begin{cases} \text{(a)} & u'=\cos^2 u+Q(x,\lambda)\sin^2 u, \\ \text{(b)} & u(a,\lambda)=u_a,\quad u(b,\lambda)=u_b+n\pi, \end{cases}$$

by the Prüfer transformation $y=Y\sin u$, $y'=Y\cos u$. It is assumed that $Q(x,\lambda)$ is continuous for $a\leqq x\leqq b$, $-\infty<\lambda<\infty$, and that it is monotonic increasing in λ. If f and F are two functions such that

$$f(x,u,\lambda)\leqq\cos^2 u+Q(x,\lambda)\sin^2 u\leqq F(x,u,\lambda),$$

for x on (a,b) and all u and λ, the inequalities maintaining on some x subinterval for all λ when u is a solution of (2a), then the systems

$$v'=f(x,v,\lambda),\quad v(a,\lambda)=u_a,$$
$$V'=F(x,v,\lambda),\quad V(a,\lambda)=u_a,$$

define respectively an under-function v and an over-function V. The roots l_n of the equation $V(b,\lambda)=u_b+n\pi$, and L_n of the equation $v(b,\lambda)=u_b+n\pi$, are bounds for the characteristic values of the problem (2) or (1), namely $l_n<\lambda_n<L_n$. By various considerations a number of sets of functions f and F and their resulting functions v and V are deduced, and the approximations to the characteristic values λ_n that are thus obtained are discussed. *R. E. Langer.*

Hulthén, Lamek. On the Sturm-Liouville problem connected with a continuous spectrum. Ark. Mat. Astr. Fys. **35A**, no. 25, 14 pp. (1948).

The equation $\phi''+k^2\phi+v(x)\phi=0$, subject to the boundary conditions $\phi(0)=0$, $\phi(\infty)$ finite, is considered. Under a restricted class of $v(x)$, the asymptotic dependence of ϕ is $\sin(kx+\eta)$. The problem is reformulated by the author so that a function of η satisfies a variational principle so that the Rayleigh-Ritz method may be applied. He uses the substitution $\phi(x)=\sin(kx+\eta)-\sin\eta y(x)$. Then $y''=k^2y=vy-v[\cos kx+\cot\eta\sin kx]$. The equivalent variational principle is $\delta J=0$ subject to the condition that N be a constant, where

$$J=\int_0^\infty[-(y')^2+k^2y^2+v(\cos kx-y)^2]dx,$$

$$N=\int_0^\infty v\sin kx(\cos kx-y)dx.$$

It is shown that the stationary values of J are given by $k[\eta_B\cot^2\eta-\cot\eta]$, where

$$\eta_B=k^{-1}\int_0^\infty v\sin^2 kxdx.$$

The method is illustrated for the case $v(x)=be^{-x}/x$.

H. Feshbach (Cambridge, Mass.).

Fröberg, Carl-Erik. Calculation of the interaction between two particles from the asymptotic phase. Physical Rev. (2) **72**, 519–520 (1947).

Preliminary note. A more detailed account is given in the paper reviewed below. *L. Hulthén* (Lund).

Fröberg, Carl-Erik. Calculation of the potential from the asymptotic phase. Ark. Mat. Astr. Fys. **34A**, no. 28, 16 pp. (1948).

Consider eigenfunctions pertaining to the continuous spectrum ($k^2\geqq 0$) of the equation

$$\{d^2/dr^2-l(l+1)/r^2+k^2+V(r)\}u=0,$$

with $u(0)=0$, the potential $V(r)$ being such as to imply $u(r)\to\sin(kr-\frac{1}{2}l+\eta(k))$, $r\to\infty$, with an $\eta(k)$ independent of r. The problem under consideration is to calculate the potential $V(r)$ if the asymptotic phase $\eta(k)$ is a known function of k, for a given l. The author finds it possible to solve the problem by means of a perturbation formalism and derives an expression for the first approximation [cf. the preliminary note reviewed above]. In an "addition in proof" a simplified formula is indicated:

$$V(r)=4(-1)^l\int_0^\infty\frac{d}{dk}(k\sin\eta(k))J_{l+1}(kr)J_{-l-1}(kr)kdk.$$

This is the inversion of the Born approximation for the phase corresponding to a known potential. Higher approximations will be treated in a subsequent paper.

L. Hulthén (Lund).

Hartman, Philip, and Wintner, Aurel. On the orientation of unilateral spectra. Amer. J. Math. **70**, 309–316 (1948).

The equation $(px')'+(q+\lambda)x=0$ is considered, where $p(s)>0$, $0\leqq s<\infty$, and $p(s)$ and $q(s)$ are real and continuous over $(0,\infty)$. The Grenzpunkt case is assumed here. It is shown that the spectrum associated with a homogeneous boundary condition at $s=0$ contains values of λ clustering at $\lambda=+\infty$. *N. Levinson* (Copenhagen).

Hartman, Philip, and Wintner, Aurel. Criteria of nondegeneracy for the wave equation. Amer. J. Math. **70**, 295–308 (1948).

The equation $x''+(f(s)+\lambda)x=0$ is considered on the half-line $s\geqq s_0$. It is assumed $f(s)$ is continuous and criteria for $f(s)$ are given which rule out the Grenzkreis case, that is, the case where all solutions $x(s)$ belong to $L^2(s_0,\infty)$. The criterion $f(s)<$constant as $s\to\infty$ was known. The following new criteria are demonstrated: (1) $f(s_2)-f(s_1)<c(s_2-s_1)$ for $s_1<s_2<\infty$; (2) $f(s)$ is monotone and $\int^\infty|f(s)|^{-\frac{1}{2}}ds=\infty$. Criteria are also given which assure that no solution of $x''+f(s)x=0$ (except $x\equiv 0$) is $L^2(s_0,\infty)$. *N. Levinson.*

Hartman, Philip. On a theorem of Milloux. Amer. J. Math. **70**, 395–399 (1948).

The linear homogeneous system $z'=A(t)z$ is considered, where z is a vector and $A(t)$ a square matrix. The author proves that if $A(t)$ is continuous for $0\leqq t<\infty$ and such that, for every solution $z(t)$, $\lim|z(t)|$ as $t\to\infty$ exists (where $|z|^2=z^*z$) then a necessary and sufficient condition for the existence of at least one solution satisfying $|z(t)|\to 0$ as $t\to\infty$ is that $\Re\int_0^t\operatorname{trace}A(s)ds\to-\infty$ as $t\to\infty$ (where $\Re$ designates real part). From this the author proves readily a theorem of Milloux for the second order equation $x''+f(s)x=0$, where $f(s)$ is a real monotone function such that $f(s)\to\infty$ as $s\to\infty$, to the effect that there is at least one nontrivial solution $x(s)$ such that $x(s)\to 0$ as $s\to\infty$.

N. Levinson (Copenhagen).

Faedo, Sandro. Proprietà asintotiche delle soluzioni dei sistemi differenziali lineari omogenei. Ann. Mat. Pura Appl. (4) **26**, 207–215 (1947).

Let A,Φ be given $n\times n$ matrices, the elements a_{ik} of A being constants and those of φ_{ik} of Φ being quasi-continuous functions of the real variable x. The system $Y'=AY$ has a fundamental set of solutions, each of which has the form $\exp(zx)P(x)$, where z is a root of the characteristic equation of A and P is a polynomial. If $\int_0^\infty x^{\nu-1}|\varphi_{ik}|dx<\infty$, where ν is the maximum multiplicity for any z, the paper establishes for $Y'=(A+\Phi)Y$ the existence of a fundamental set of solutions each member of which has the form $\exp(zx)[P(x)+x^{j-1}Q]$, where for each z of multiplicity m the index $j=1,\cdots,m$, and where each Q is a series of continuous functions determined recursively by quadratures and converging absolutely and uniformly for $x_0\leqq x$.

J. M. Thomas (Durham, N. C.).

Teofilato, Pietro. Sopra alcuni sistemi differenziali a soluzioni sensibilmente costanti. Pont. Acad. Sci. Acta **8**, 119–130 (1944).

The author studies a system $dx_i/dt=\epsilon_i+k\sum_{j=1}^n\lambda_{ij}x_j$ ($i=1,\cdots,n$), assuming either that $k=1$ and that the ϵ's and λ's are slowly varying functions of t, or that the ϵ's and λ's are general functions of t and that the parameter k can assume arbitrarily large values. He also assumes that $\lambda_{11}+\lambda_{22}+\cdots+\lambda_{nn}=0$; that the matrices

$$\begin{pmatrix}\lambda_{11}&\cdots&\lambda_{1n}&\epsilon_1\\ \cdots&&\cdots&\\ \lambda_{n1}&\cdots&\lambda_{nn}&\epsilon_n\end{pmatrix},\qquad\begin{pmatrix}\lambda_{11}&\cdots&\lambda_{1n}\\ \cdots&&\cdots\\ \lambda_{n1}&\cdots&\lambda_{nn}\end{pmatrix}$$

have the same rank $n-m$; and that the secular equation

$$\begin{vmatrix} \lambda_{11}-u & \cdots & \lambda_{1n} \\ \cdots & & \cdots \\ \lambda_{n1} & \cdots & \lambda_{nn}-u \end{vmatrix} = 0$$

has a root $u=0$ of multiplicity m, and $n-m$ distinct pure imaginary roots which are mutually conjugate in pairs. The following results are obtained. (A) If $k=1$, and the λ's and ϵ's are bounded slowly varying functions, the x's are bounded functions. Also, if $n-m=2$, the x's may remain in the neighborhoods of their respective initial values, despite the adiabatic variations of the λ's and the ϵ's. (B) If the λ's and ϵ's are functions of t which satisfy Lipschitz conditions, then, as k increases without limit, the x's tend toward functions having the properties stated under (A) above. *L. A. MacColl* (New York, N. Y.).

Letov, A. M. On the theory of an isodromic regulator. Akad. Nauk SSSR. Prikl. Mat. Meh. **12**, 363–368 (1948). (Russian)

The circuit equations lead to nonlinear differential equations where the nonlinear terms have small parameters as coefficients. The Poincaré method is applied to obtain periodic solutions. *R. Bellman.*

Bellman, Richard. On an application of a Banach-Steinhaus theorem to the study of the boundedness of solutions of non-linear differential and difference equations. Ann. of Math. (2) **49**, 515–522 (1948).

This paper deals with systems $\dot{z}=A(t)z+\phi(z,t)$ of nonlinear differential equations, where $A(t)$ is a continuous matrix, $t\geqq 0$, and z and ϕ are vectors, and with systems $\dot{z}=A(t)z+\phi(t)$ of linear differential equations. The matrix function $A(t)$ is fixed and the same for both systems, but the functions $\phi(z,t)$, $\phi(t)$, respectively, can vary over certain classes C_1, C_2, respectively. The author proves theorems of the following type. If all the solutions of the linear system for any $\phi(t)\in C_2$ are bounded (i.e., each component is bounded) then all the solutions of the nonlinear system for any $\phi(z,t)\in C_1$ are also bounded. The different theorems are obtained by taking different pairs of classes C_1, C_2: C_1, C_2 the classes of bounded functions, or C_1, C_2 the classes of functions bounded in norm by some function $f(t)$ which is integrable from 0 to ∞, or also $f(t)$ square integrable from 0 to ∞. In the first case, the result had already been proved by Perron [Math. Z. **32**, 465–473 (1930)]. The extension to the other cases is new. The method is different from the one used by Perron and more direct. It is shown that in all cases the results can be derived quickly from a modification of a result of Banach and Steinhaus [cf. Zygmund, Trigonometric Series, Warsaw, 1935, p. 99 (iv)]. Further boundedness results are obtained if $\phi(z,t)$ satisfies inequalities of the type $\|\phi(z,t)\|\leqq\|z\|\cdot f(t)$, provided the norm of the initial condition is sufficiently small. Finally the author outlines how the method can be used for difference equations. *F. Bohnenblust* (Pasadena, Calif.).

Cartwright, M. L. Topological aspect of forced oscillations. Research **1**, 601–606 (1948).

Expository article.

Cartwright, M. L., and Littlewood, J. E. Errata: On non-linear differential equations of the second order. II. Ann. of Math. (2) **49**, 1010 (1948).

The paper appeared in the same Ann. (2) **48**, 472–494 (1947); these Rev. **9**, 35.

Cafiero, F. Un'osservazione sulla continuità rispetto ai valori iniziali degli integrali dell'equazione: $y'=f(x,y)$. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **3**, 479–482 (1947).

The author proves the following result. Any group of conditions sufficient to assure the existence and uniqueness, with respect to the initial values, of the integral of the equation $y'=f(x,y)$ is also sufficient to assure the continuous dependence of the solution on the initial values. *R. Bellman* (Stanford University, Calif.).

Zwirner, Giuseppe. Un criterio d'esistenza e d'unicità per gli integrali dell'equazione $y'=\lambda f(x,y)$ passanti per due punti assegnati. Boll. Un. Mat. Ital. (3) **3**, 15–18 (1948).

The author states the following theorem, and sketches the proof briefly. Let the function $f(x,y)$ be defined and continuous in the rectangle R: $x_0\leqq x\leqq x_1$, $a\leqq y\leqq b$; let $\partial f/\partial y$ exist and be continuous in R; and let y_0 and y_1 be two numbers such that $a\leqq y_0<y_1\leqq b$. Then the problem $dy/dx=\lambda f(x,y)$, $y(x_0)=y_0$, $y(x_1)=y_1$ has a unique solution $[\lambda, y(x)]$ if there exist two nonnegative summable functions $p_1(x)$, $p_2(x)$ which are defined in $x_0\leqq x\leqq x_1$, and which have the following properties:

$$\int_{x_0}^{x_1} p_1(x)dx>0;$$

$$p_1(x)\leqq f(x,y)\leqq p_2(x), \qquad (x,y)\in R;$$

$$|y_1-y_0|\int_{x_0}^{x_1} p_2(x)dx<(b-y_0)\int_{x_0}^{x_1} p_1(x)dx;$$

$$|y_1-y_0|\int_{x_0}^{x_1} p_2(x)dx<(y_0-a)\int_{x_0}^{x_1} p_1(x)dx.$$

L. A. MacColl (New York, N. Y.).

Szarski, J. Sur un système d'inégalités différentielles. Ann. Soc. Polon. Math. **20** (1947), 126–134 (1948).

Let $f_i(x, y_1, \cdots, y_n)$ be continuous functions on an open domain and let $\Phi_i(x)$, $i=1, \cdots, n$, be locally absolutely continuous functions such that $\Phi_i'(x)\leqq f_i(x, \Phi_1(x), \cdots, \Phi_n(x))$ almost everywhere on $x_0\leqq x<x_0+a$. Then $\Phi_i(x_0)\leqq\Psi_i(x_0)$ implies $\Phi_i(x)\leqq\Psi_i(x)$ on $x_0\leqq x<x_0+a$, where $y_i=\Psi_i(x)$ is the maximal solution of $y_i'=f_i$ through the point $(x_0, \Psi_1(x_0), \cdots, \Psi_n(x_0))$. The author makes the superfluous assumption that each function f_i is nondecreasing with respect to each of the variables y_j. *P. Hartman.*

Szarski, J. Sur une propriété asymptotique des intégrales d'un système d'équations différentielles ordinaires. Ann. Soc. Polon. Math. **20** (1947), 161–168 (1948).

The result proved can be illustrated by the following. Let $f_i(x, y_1, \cdots, y_n)$ be continuous functions defined on the set E: $0\leqq x<\infty$, $\sum_{i=1}^{n} y_i^2\leqq\gamma(x)$, where $\gamma(x)$ is positive and continuous, and having the property that a solution of the system $y_i'=f_i$ through a point $(x_0, y_1^0, \cdots, y_n^0)$, $x_0>0$, of E possesses a continuation over the interval $0\leqq x\leqq x_0$. Then the set of points $(0, y_1, \cdots, y_n)$ of E reached by at least one solution which is defined for $0<x<\infty$ is a nonempty closed connected set. *P. Hartman* (Baltimore, Md.).

Ważewski, T., et Szarski, J. Sur l'unicité des intégrales de l'équation de Clairaut, modifiée. Ann. Soc. Polon. Math. **20** (1947), 157–160 (1948).

The authors prove that if $f(x,y)$ is continuous in a rectangle R and has the property that, through every point

of R, there passes a line segment which is an integral of the differential equation $y'=f(x, y)$, then these line segments are the only integrals of the differential equation. This result is contained in a general theorem of E. R. van Kampen [Amer. J. Math. 59, 144–152 (1937)]; van Kampen's theorem also answers affirmatively the question raised by the authors concerning the possibility of extending the above uniqueness theorem to systems. *P. Hartman.*

Ważewski, Tadeusz. Sur un principe topologique de l'examen de l'allure asymptotique des intégrales des équations différentielles ordinaires. Ann. Soc. Polon. Math. 20 (1947), 279–313 (1948).

Verfasser gibt Verallgemeinerungen und Anwendungen eines schon von ihm ausgesprochenen und bewiesenen Satzes [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 3, 210–215 (1947); diese Rev. 9, 512]. *G. Scorza Dragoni* (Padova).

Saltykow, N. Étude sur les intégrales singulières des équations différentielles. Acad. Serbe. Bull. Acad. Sci. Mat. Nat. A. 7, 89–134 (1941).

Let $f(x, y, y')$ have derivatives with respect to its arguments, and assume that the equation (*) $f(x, y, dy/dx)=0$ has a general integral (**) $y=V(x, C)$. If (*) has a singular integral representing the envelope of the curves (**), the paper shows this singular integral is a solution of the Lagrange equations associated with (*). Conversely, it is also shown that a solution of the associated Lagrange equations is either a particular solution of (*) or an envelope of (**). The above results are extended to first order equations $f(x_1, \cdots, x_n, z, p_1, \cdots, p_n)=0$, where $p_i=\partial z/\partial x_i$. [In the next to the last paragraph on page 94 replace y' by y. The only way the reviewer could justify the statement made in this paragraph was to replace $[\partial^2 V/\partial x\partial C]=0$ by $\partial^2 V/\partial x\partial C=0$.] *F. G. Dressel* (Durham, N. C.).

Saltykow, N. Transformations tangentielles linéaires. Acad. Serbe. Bull. Acad. Sci. Mat. Nat. A. 7, 71–87 (1941).

Let $x_1, \cdots, x_n$ and $x_1', \cdots, x_n'$ denote the old and new independent variables while z and z' denote the old and new dependent variables, respectively. A tangential transformation is called linear if it can be expressed in the form $x_i'=\sum_{k=1}^{n} a_{ik}p_k+c_iz+d_i$, $p_k=\partial z/\partial x_k$, $i=1, \cdots, n$, $z'=\sum_{k=1}^{n} b_kp_k+mz+r$, where $a_{ij}, c_i, d_i, b_k, m, r$ are functions of $x_1, \cdots, x_n$. The present paper presents methods of forming linear tangential transformations. *F. G. Dressel.*

Saltykow, N. Équations aux dérivées partielles du premier ordre intégrables par séparation des variables. Acad. Serbe. Bull. Acad. Sci. Mat. Nat. A. 7, 135–159 (1941).

Assuming that $H(x_1, \cdots, x_n; p_1, \cdots, p_n)=0$, $p_i=\partial z/\partial x_i$, can be integrated by the method of separation of variables, the author presents some methods that may help one determine the possibility of this separation property. *F. G. Dressel* (Durham, N. C.).

Saltykow, N. Équations aux dérivées partielles d'ordre supérieur réductibles aux celles du premier ordre. Acad. Serbe. Bull. Acad. Sci. Mat. Nat. A. 7, 161–190 (1941).

Let x, y be the independent variables and z the dependent variable, and define $z_0=z$,

$$z_k=P(x, y)\partial z_{k-1}/\partial x+Q(x, y)\partial z_{k-1}/\partial y,$$

$k=1, \cdots, n$. The author points out that the general integral of a partial differential equation of order n of the form $z_n=f(x, y, z, z_1, \cdots, z_{n-1})$ can be obtained by making use of the ordinary differential equations

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz_{n-1}}{f}=\frac{dz_{n-2}}{z_{n-1}}=\cdots=\frac{dz_0}{z_1}.$$

Most of the paper is devoted to applications to second, third and fourth order differential equations. *F. G. Dressel* (Durham, N. C.).

Saltykow, N. Application des invariants différentiels à l'intégration des équations aux dérivées partielles du premier ordre à une fonction inconnue. Acad. Serbe. Sci. Publ. Inst. Math. 1, 21–48 (1947).

Assume the linear partial differential forms

$$X(f)\equiv\sum_{i=1}^{n} X_i(x_1, \cdots, x_n)\frac{\partial f}{\partial x_i};\quad Y^k(f)\equiv\sum_{i=1}^{n} Y_i^k(x_1, \cdots, x_n)\frac{\partial f}{\partial x_i},$$

where $k=1, \cdots, s$, are such that

$$(*)\qquad X(Y^k(f))-Y^k(X(f))\equiv 0,\qquad k=1, \cdots, s.$$

If $\varphi(x_1, \cdots, x_n)$ is a solution of the system $Y^k(f)=0$, then from (*) the functions $\varphi_1\equiv X(\varphi), \varphi_2\equiv X(\varphi_1), \cdots, \varphi_{\nu+1}\equiv X(\varphi_\nu)$ are seen to be solutions of $Y^k(f)=0$. If $\varphi, \varphi_1, \cdots, \varphi_\nu$ are independent functions and $\varphi_{\nu+1}=\theta(\varphi, \varphi_1, \cdots, \varphi_\nu)$, the author points out that $\varphi, \varphi_1, \cdots, \varphi_\nu$ may be used as follows to solve the system $X(f)=0$, $Y^k(f)=0$. Assume $f=\Phi(\varphi, \varphi_1, \cdots, \varphi_\nu)$; then the $Y^k(f)=0$ are satisfied and $X(f)=0$ takes the form

$$\varphi_1\frac{\partial\Phi}{\partial\varphi}+\varphi_2\frac{\partial\Phi}{\partial\varphi_1}+\cdots+\varphi_\nu\frac{\partial\Phi}{\partial\varphi_{\nu-1}}+\theta\frac{\partial\Phi}{\partial\varphi_\nu}=0.$$

Any solution of this last equation leads to solutions of the system $Y^k(f)=0$ enlarged by $X(f)=0$. The paper also outlines a similar method for integrating systems of nonlinear partial differential equations of the first order. *F. G. Dressel* (Durham, N. C.).

Petrescu, Şt. Sur les éléments singuliers et sur la géométrisation des systèmes de deux équations de Pfaff à huit variables. Bull. École Polytech. Jassy [Bul. Politehn. Gh. Asachi. Iaşi] 2, 290–340 (1947).

The system S studied can be written $\omega_1\equiv a_idx^i=0$, $\omega_2\equiv b_idx^i=0$, where $i=1, \cdots, 8$, where a, b are given functions of $x^1, \cdots, x^8$ and where the summation convention is employed. The treatment centers about the pencil of forms $a\omega_1'+\beta\omega_2'$ mod (ω_1, ω_2) and the rank of $\det(aa_{ij}+\beta b_{ij})$, where ω_1', ω_2' are the derived forms and a_{ij}, b_{ij} are their coefficients. The system is solved for dx^7, dx^8 and the geometrical language of a projective space of five dimensions with homogeneous coordinates $dx^1, \cdots, dx^6$ is employed. The families of singular elements are defined and employed in reducing S to special forms. *J. M. Thomas.*

Castoldi, L. Attorno a una dimostrazione geometrica di un teorema di analisi. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 3, 496–498 (1947).

The known condition that $X(x, y, z)dx+Y(x, y, z)dy+Z(x, y, z)dz$ admit an integrating factor is proved necessary and sufficient by vectorial methods. *J. M. Thomas* (Durham, N. C.).

Ehresmann, Charles, et Libermann, Paulette. Sur les formes différentielles extérieures de degré 2. C. R. Acad. Sci. Paris 227, 420–421 (1948).

Let Ω denote a quadratic differential form which involves $2n$ variables and which has rank $2n$ under Grassmann multi-

plication of its differentials. The present note communicates the following results: if $2<n$ and the derived form Ω' satisfies $\Omega'=\theta\Omega$, where θ is linear, Ω can be written locally $dx_1dy_1+\cdots+dx_ndy_n$ or $y_1(dx_1dy_1+\cdots+dx_ndy_n)$ according as $\Omega'=0$ or $\Omega'\neq 0$; if $2=n$, then $\Omega'=\theta\Omega$ and in general linear forms $\omega_1, \omega_2, \pi_1, \pi_2$ can be determined so that $\Omega=\omega_1\pi_1+\omega_2\pi_2$, $\theta'=\lambda(\omega_1\pi_1-\omega_2\pi_2)$, $\theta=\pi_1+\pi_2$. *J. M. Thomas.*

Backes, F. Sur la notion d'intégrale complète. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **34**, 373–388 (1948).

Let the equation $F(x_1, \cdots, x_n, z, p_1, \cdots, p_n)=0$, $p_i=\partial z/\partial x_i$, have (**) $z=w(x_1, \cdots, x_n; a_1, \cdots, a_n)$ as a solution depending on the n arbitrary constants a_i. If the $(n+1)\times n$ matrix whose ith row is $\partial w/\partial a_i, \partial^2 w/\partial x_1\partial a_i, \cdots, \partial^2 w/\partial x_n\partial a_i$, has rank n, the paper shows that the elimination of the a_i between (*) and $p_i=\partial w/\partial x_i$ leads to a function $\Phi(x_1, \cdots, x_n, z, p_1, \cdots, p_n)$ which is equivalent to F in the sense that the vanishing of Φ implies the vanishing of F and conversely.

F. G. Dressel (Durham, N. C.).

Caianiello, Eduardo. Il metodo di Mayer e l'integrazione dei sistemi completi di equazioni lineari alle derivate parziali del primo ordine nel campo reale. Giorn. Mat. Battaglini (4) **77**, 164–171 (1947).

Mayer's method is applied to prove the existence of the solution of a complete system of linear homogeneous partial differential equations of the first order whose coefficients are real functions of class C'' in the real independent variables. *J. M. Thomas* (Durham, N. C.).

Ascoli, Guido. Sopra i sistemi lineari isotropi e le loro proprietà integrali. Pont. Acad. Sci. Comment. **7**, 207–281 (1943).

The author gives a general theory of linear partial differential equations with spherical symmetry. Let (F) be a system of functions $f(x_1, \cdots, x_n)$ defined in a spherical shell $r'\leqq r\leqq r''$, where r is the distance of the point $(x_1, \cdots, x_n)$ from the origin. Then (F) is called an "isotropic" system if it contains with every f all functions obtained from f by sense-preserving orthogonal substitutions. Of special interest are the "linear isotropic systems." The author determines all linear isotropic systems which form a finite-dimensional linear space. The most general such system is shown to consist of the functions of the form $\sum_{i,s}\psi_s^i(r)Y_s(P)$, where $\psi_s^i(r)$ are any fixed functions, finite in number, and the $Y_s(P)$ are any arbitrary spherical harmonics of degree s. (Here a spherical harmonic of degree s shall mean a function defined in the points P of the unit sphere, whose values coincide with those of a form of degree s which satisfies Laplace's equation.) A linear isotropic system on the unit sphere coincides with the spherical harmonics of some fixed degree s if it is finite-dimensional and "irreducible," i.e., contains no nontrivial subsystem forming a linear isotropic system. In this way the author obtains a characterization of spherical harmonics without reference to the potential equation.

A linear partial differential equation is called "isotropic" if it is invariant under all sense-preserving orthogonal transformations. The most general such equation is shown to be of the form $E[u]=R(D, \Delta)u=0$. Here D denotes the radial differentiation $D=\partial/\partial r$, Δ the Laplace operator, and R any polynomial with coefficients depending on r only. The solutions of a linear isotropic equation will form a linear isotropic system. The isotropic operator E will transform a linear isotropic system into a linear isotropic system. For any isotropic operator E and any spherical harmonic $Y_s(P)$ of degree s, one has (1) $E[g(r)Y_s(P)]=L_s(g)Y_s(P)$, where L_s is an ordinary differential operator which depends only on E and s. The "separation of variables" in the operator E expressed by formula (1) can be seen to be characteristic for isotropic operators E. Formula (1) can also be used to characterize spherical harmonics by means of arbitrary isotropic operators, without reference to the special potential equation.

The author applies his results to a number of special isotropic differential equations, and obtains in this way a number of classical mean value theorems and series expansions for the solutions. *F. John* (New York, N. Y.).

Lykov, A. V. On the theory of thermal waves. Izvestiya Akad. Nauk SSSR. Otd. Tehn. Nauk **1948**, 1003–1008 (1948). (Russian)

Through the Laplace transformation, the author derives formally the temperature in an infinite plate of finite thickness, initially at temperature zero, whose faces are exposed to a medium whose temperture is oscillatory. Newton's law of surface heat transfer is assumed to hold at the faces. When conditions are specialized, he finds the known formula for temperature in a semi-infinite solid [Carslaw and Jaeger, Conduction of Heat in Solids, Oxford, 1947, p. 47; these Rev. **9**, 188]. Results are also stated for the cylinder of infinite length and for the sphere. *R. E. Gaskell.*

Prinetti, Tommaso. Sulle sorgenti mobili di calore. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **81–82**, 182–188 (1948).

After showing why one may look on the function

$$(*)\qquad u(x, t)=\int_{t_0}^{t}\frac{q(\tau)}{(t-\tau)^{\frac{1}{2}}}\exp\left\{-\frac{[x-f(\tau)]^2}{4(t-\tau)}\right\}d\tau$$

as the heat generated by a source moving from time t_0 to time t by the law $\xi=f(\tau)$, and of intensity proportional to $q(\tau)$ at time τ, the author discusses several different physical problems that can be solved by making use of the above interpretation of (*). *F. G. Dressel.*

Vodicka, V. Der Kreiszylinder in einem zeitlich veränderlichen Temperaturfelde. Schweiz. Arch. Angew. Wiss. Tech. **14**, 177–180 (1948).

For the cylinder S: $-z_0<z<z_0$, $0\leqq r<r_0$ the following heat conduction problem is treated. Find u such that $u_t=a^2(u_{rr}+r^{-1}u_r+u_{zz})$, (r, θ, z) in S; $u_z\pm h(u-Ae^{i\omega t})=0$ for $z=\pm z_0$; $u_r+h(u-Ae^{i\omega t})=0$ for $r=r_0$ (a, h, A, v, z_0, r_0 being constants). The formal infinite series solution is obtained by the method of separation of variables.

F. G. Dressel (Durham, N. C.).

Ward, A. G. The diffusion of decomposition products through plastic materials. Philos. Mag. (7) **39**, 621–632 (1948).

Mathematically, the following boundary value problems are treated by the method of separation of variables. Let R be a region, D and Q constants; the function u (concentration of gas in R) is to satisfy $u_t=D(u_{xx}+u_{yy}+u_{zz})+Q$ in R (the stationary state $u_t=0$ is also treated). On the boundary of R, u is assumed to be constant. The types of regions R treated are those bounded by two parallel planes, or by one (or two) cylinder(s), or by a spherical surface.

F. G. Dressel (Durham, N. C.).

Titchmarsh, E. C. Analysis for physicists. J. London Math. Soc. **23**, 69–79 (1948).

This is a presidential address before the London Mathematical Society. The author points out that there are many interesting problems in fractional operations and expansion theorems which arise in theoretical physics and which should attract the mathematician, especially the analyst.

A. E. Heins (Pittsburgh, Pa.).

Caldirola, Piero, e Sillano, Pietro. Integrazione dell'equazione delle onde sferiche smorzate e forzate col calcolo operatorio. Pont. Acad. Sci. Acta **10**, 229–242 (1946).

A solution is provided for the inhomogeneous wave equation:

$$\partial^2 u/\partial t^2 - \Delta_3 u - ku = X,$$

where X is a known function of x, y, z and Δ_3 is the three dimensional Laplacian operator in rectangular coordinates. The solution is expressed in terms of initial value data, i.e., $u(0, x, y, z)$ and $u_t(0, x, y, z)$ and boundary value data, i.e., u and $\partial u/\partial n$ on a given surface. The time part of the problem (the t variable) is handled with the unilateral Laplace transform and the spatial part by an application of Green's theorem with a free space Green's function kernel. There is extraneous boundary value information in the solution which theoretically could be eliminated.

A. E. Heins (Pittsburgh, Pa.).

Agostinelli, Cataldo. Sul problema di Cauchy per l'equazione differenziale delle piastre vibranti. Ann. Mat. Pura Appl. (4) **26**, 27–41 (1947).

The solution $u(x, y, t)$ of the following boundary value problem is given in the form of a definite double integral: $\Delta\Delta u + \alpha^2 \partial^2 u/\partial t^2 = 0$, $t > t_0$, $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$, α a constant; $u(x, y, t_0) = \varphi(x, y)$, $u_t(x, y, t_0) = \theta(x, y)$. The assigned functions φ, θ are assumed to have derivatives of all orders.

F. G. Dressel (Durham, N. C.).

Teofilato, Pietro. Integrazione di una classe di equazioni a derivate parziali del secondo ordine. Pont. Acad. Sci. Comment. **8**, 35–60 (1944).

The author employs the "symbolic operational calculus" to treat the integration of the partial differential equation

$$(0) \qquad y_{tt} + a(t)y_t + b(t)y = c(t)y_{xx} + \varphi(t, x),$$

subject to the initial conditions

$$y(0, x) = \sum_1^\infty b_n \sin n\pi x/l; \quad y_t(0, x) = \sum_1^\infty g_n \sin n\pi x/l,$$

with the sequences b_n and g_n of the order of $1/n^2$, and the end conditions $y(t, 0) = 0$, $y(t, l) = 0$. By means of the substitution $y(t, x) = z(t, x) \exp\{-\frac{1}{2}\int_0^t a(s)ds\}$, the equation

$$(1) \qquad [\partial^2/\partial t^2 + (B - c\Delta^2)]z(t, x) = \psi(t, x)$$

is obtained, where $B = b - \frac{1}{4}a^2 - \frac{1}{2}da/dt$,

$$\psi(t, x) = \exp\{\tfrac{1}{2}\textstyle\int_0^t a(s)ds\} \cdot \varphi(t, x),$$

and $\Delta = \partial/\partial x$. Treating Δ as a constant, (1) is factored [see G. Mammana, Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (6) **9**, 538–544, 608–615 (1929)]:

$$(\partial/\partial t + \alpha(t, \Delta))(\partial/\partial t + \beta(t, \Delta))z(t, x) = \psi(t, x).$$

Formally,

$$z(t, x) = (\partial/\partial t + \beta(t, \Delta))^{-1}(\partial/\partial t + \alpha(t, \Delta))^{-1}\psi(t, x),$$

and it remains only to evaluate α and β, and to apply the inverse operators to ψ in order to obtain a formal solution of (1). The calculation of α and β leads to the consideration of the auxiliary ordinary differential equation of (0): $L(v) \equiv v_{tt} + (B - c\Delta^2)v = 0$, whose general solution is

$$v(t, \Delta) = H(\Delta)w(t, \Delta) + K(\Delta)\Phi(w(t, \Delta)),$$

where w is a particular solution,

$$\Phi(w(t, \Delta)) = w(t, \Delta)\int_T^t \{1/w^2(s, \Delta)\}ds,$$

and T is an arbitrary constant. The formal general solution of (1), still in operational form, is

$$z(t, x) = w(t, \Delta)\Gamma_1(x) + \Phi(w(t, \Delta))\Gamma_2(x) + \Phi(w(t, \Delta))\int_T^t w(s, \Delta)\psi(s, x)ds - w(t, \Delta)\int_T^t \Phi(w(s, \Delta))\psi(s, x)ds,$$

where Γ_1 and Γ_2 are arbitrary functions. Admitting that the operators in question are distributive, it is only necessary to compute the result of operating with them on the function $n\pi x/l$. Once this is done, the formal solution y of (0) is explicitly written out and the unknown Fourier coefficients appearing in its expansion are determined by means of f_n and g_n. Assuming the Fourier series solution so obtained to be termwise differentiable, etc., it is found that it satisfies (0) and the other prescribed conditions. As a sort of check, the solution z of (1) found is shown to reduce to $F(x + \sigma t) + G(x - \sigma t)$, the known general solution of d'Alembert for the wave equation, when $\psi(t, x) \equiv 0$, $a(t) \equiv 0$, $b(t) \equiv 0$, $c(t) = \sigma^2 = \text{constant}$. The convergence of the solution is ascertained in the special case in which the coefficients of (0) satisfy $b - \frac{1}{4}a^2 - \frac{1}{2}da/dt = ge^{kt}$, $c = le^{kt}$, with g, l and k constants, and $k > 0$.

J. B. Diaz (Providence, R. I.).

Teofilato, Pietro. Integrazione di una classe di equazioni alle derivate parziali del quarto ordine e applicazione allo studio delle vibrazioni di una trave. Pont. Acad. Sci. Comment. **8**, 63–75 (1944).

The author has applied the "symbolic operational calculus" to a class of partial differential equations of the second order [see the preceding review]. The same calculus is now employed to obtain a formal solution of the fourth order equation

$$y_{tt} + a(t)y_t + b(t)y + A_2(t)y_{xx} + A_4(t)y_{xxxx} = \varphi(t, x),$$

subject to the initial conditions $y(0, x) = \sum_1^\infty b_n \sin n\pi x/l$, $y_t(0, x) = \sum_1^\infty g_n \sin n\pi x/l$, with the sequences f_n and g_n of the order of n^{-2}, and the end conditions $y(t, 0) = y_{xx}(t, 0) = 0$, $y(t, l) = y_{xx}(t, l) = 0$. This result is then applied to the special case of a simply supported beam acted upon by a moving load. Here $\mu/\rho = a$, $P/\rho = b$, $0 = A_2$, and $EI/\rho = A_4$, where ρ, E, I, P and μ are constants, and φ is a certain Fourier sine series. It is shown that in this particular case the formal series solution is convergent.

J. B. Diaz.

Bruk, S. Z. The fundamental solutions of a system of differential equations of parabolic type. Doklady Akad. Nauk SSSR (N.S.) **60**, 9–12 (1948). (Russian)

The fundamental solutions of the following system of equations are studied:

$$\frac{\partial u_i}{\partial t} - \sum_{j; k_1, \cdots, k_n} A_{ij}^{k_1 \cdots k_n}(t; x_1, \cdots, x_n)\frac{\partial^M u_j}{\partial x_1^{k_1} \cdots \partial x_n^{k_n}} = 0$$

$(i, j=1, \cdots, N; \sum_{r=1}^{n} k_r = M)$, where the characteristic roots of the λ-matrix

$$\lambda E - \| \sum_{k_1, \cdots, k_n} A_{ij}^{k_1 \cdots k_n}(t; x_1, \cdots, x_n)(i\alpha_1)^{k_1} \cdots (i\alpha_n)^{k_n} \|$$

satisfy $\Re\lambda(\alpha_1, \cdots, \alpha_n; t; x_1, \cdots, x_n) < -\delta < 0$ for all x_i, for $t_0 \leqq t \leqq T$ and for all real α_i such that $\sum \alpha_i^2 = 1$.

The solutions are found, following a method by E. E. Levi. In the original equations the x_i inside the parentheses are replaced by parameters y_i which are independent of the x_i. The system of equations so obtained has coefficients which depend on t only and its fundamental solutions may be constructed by Fourier analysis. If these solutions are denoted by $u_r^s(t; t-\tau; y; x-\xi)$, the fundamental solution of the original equations is

$$z_r^s(t; \tau; x; \xi) = u_r^s(t; t-\tau; x; x-\xi) + \int_\tau^t dt' \int_{-\infty}^{\infty} \sum_r u_j^r(t; t-t'; x; x-x') \varphi_r^s(t', \tau; x'; \xi) dx_1' \cdots dx_n'$$

$(j, s, r = 1, \cdots, N)$. The functions φ_r^s will have to be determined from integral equations. That solutions of these integral equations exist is claimed in a lemma. Proofs of this and other essential lemmas are not given.

P. A. Lagerstrom (Pasadena, Calif.).

Difference Equations, Special Functional Equations

Montel, Paul. Sur un système d'équations fonctionnelles. Ann. Soc. Polon. Math. **21**, 99–106 (1948).

Proofs of results announced previously [C. R. Acad. Sci. Paris **226**, 1053–1055 (1948); these Rev. **9**, 515].

R. P. Boas, Jr. (Providence, R. I.).

Stephens, C. F. Nonlinear difference equations analytic in a parameter. Trans. Amer. Math. Soc. **64**, 268–282 (1948).

Among the problems considered in this paper is the system of difference equations

$$y_i(x+1) = f_i\{y_1(x), y_2(x), \cdots, y_n(x); P(x); x\},$$

$f_i\{0, \cdots, 0; 0; x)\} \equiv 0$, $i = 1, 2, \cdots, n$, where $P(x)$ represents a parameter that is independent of the $y_i(x)$, and is periodic in x of period unity. The f_i are continuous in x and bounded for all $y_i(x)$, $P(x)$, and x in the domain G: $|y_i(x)| \leqq r_i$, $|P(x)| \leqq \rho$, $|x| \geqq R$. Furthermore, the f_i are analytic functions of the $y_i(x)$ and $P(x)$ in G, and can be developed into convergent power series in the variables $P(x)$ and $y_i(x)$ starting with terms of the second degree. A formal solution is obtained in the form $y_i(x) = \sum_{k=2}^{\infty} C_i^{(k)}(x) P^k(x)$, where the $C_i^{(k)}(x)$ are given explicitly; these functions $y_i(x)$ are shown to be continuous functions of x and analytic functions of $P(x)$ in a certain domain.

D. Moskovitz.

Wright, E. M. Linear difference-differential equations. Proc. Cambridge Philos. Soc. **44**, 179–185 (1948).

The author considers the general linear differential-difference equation $\sum_{\mu=0}^{m} \sum_{\nu=0}^{n} A_{\mu\nu}(x) y^{(\nu)}(x+b_\mu) = v(x)$, where x is a real variable, $v(x)$, $A_{\mu\nu}(x)$ are known, $0 = b_0 < b_1 < \cdots < b_m$, and the boundary conditions are of the type: $y^{(\nu)}(x_0)$ is given for $\nu \leqq n-1$, and the value of $y^{(n)}(x)$ is known in some interval $x_0 \leqq x < x_0 + b_m$. By means of elementary methods it is shown that some slight restrictions on the coefficients suffice to determine bounds on $y^{(n)}(x)$ as $x \to +\infty$. As the author points out, results of this nature are a necessary preliminary to a more detailed development of the theory using transform methods.

R. Bellman.

Dwinas, S. A deduction of the Laplace-Gauss law of errors. Revista Mat. Hisp.-Amer. (4) **8**, 12–18 (1948). (Spanish)

It is proved that the normal density function $f(x)$ is the only density for which $f(x)f(y) = \phi((x^2+y^2)^{\frac{1}{2}})$, and that $a e^{-a|x|}/2$ is the only density for which $f(x)f(y) = \phi(|x|+|y|)$.

W. Feller (Ithaca, N. Y.).

Integral Equations

Richard, Ubaldo. Rapporti tra le equazioni di Volterra e le serie di polinomi di Laguerre. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **81–82**, 316–331 (1948).

This paper gives the solution of integral equations with variable upper limit and kernel of the particular type $K(x-y)$ with the aid of Laguerre series of order zero. For the equation of the second kind the Fourier-Laguerre coefficients c_n of the unknown function are expressed in terms of the coefficients a_n, b_n of the given function and of the kernel as follows:

$$c_n - \sum_{k=0}^{n} c_k (b_{n-k} - b_{n-k-1}) = a_n.$$

The equation of the first kind is reduced to that of the second kind. The conditions of convergence of the solution $\sum_0^\infty c_n L_n(x)$ are studied thoroughly.

E. Kogbetliantz.

Unsöld, A. Über die Integralgleichung des Strahlungsgleichgewichtes in Sternatmosphären. Z. Astrophys. **24**, 363–383 (1948).

The radiative equilibrium of a scattering and absorbing atmosphere with scattering and absorption coefficients dependent on the frequency is considered by deriving an integral equation analogous to the Schwarzschild-Milne integral equation for a gray atmosphere and then seeking a solution of this equation by iterating a solution which has the correct asymptotic behaviour at infinity. The application of the method to the case when the absorption coefficient k, has one of two values k_1 and k_2 with probabilities a_1 and a_2 ($a_1 + a_2 = 1$) is considered in detail, and the solution obtained by the iterative procedure is compared with the earlier ones of Chandrasekhar [Monthly Not. Roy. Astr. Soc. **96**, 21–42 (1935)] and Hopf [ibid., 522–533 (1936)].

S. Chandrasekhar (Williams Bay, Wis.).

Gratton, L. Soluzione dell'equazione del trasporto nel caso cilindrico. Nuovo Cimento (9) **5**, 68–77 (1948).

The equation of transfer

$$\mu \frac{\partial \psi}{\partial r} + \frac{\sin^2 \varphi}{r} \frac{\partial \psi}{\partial \mu} - \frac{\mu \sin \varphi \cos \varphi}{r} \frac{\partial \psi}{\partial \varphi} = -\psi(r, \mu, \varphi) + (4\pi)^{-1} \int_0^{2\pi} d\varphi' \int_{-1}^{1} d\mu' \psi(r, \mu', \varphi')$$

is solved by expanding ψ in spherical harmonics in the form

$$\psi(r, \mu, \varphi) = \sum_{l=0}^{\infty} \sum_{m=0}^{l} \psi_{l,m} P_l^m(\mu) \cos m\varphi.$$

The solutions in the "first approximation" including the terms ψ_{00}, ψ_{10}, ψ_{11}, ψ_{20}, and in the second approximation

when the additional terms ψ_{21}, ψ_{22}, ψ_{20}, ψ_{21} and ψ_{20} are also included, are worked out explicitly. As is to be expected the solution in the second approximation involves quadratures over Bessel functions. *S. Chandrasekhar.*

Mark, J. C. The spherical harmonic method. I. (General development of the theory). National Research Council of Canada. Division of Atomic Energy. Document no. **CRT-340** (N.R.C. **1588**), i+23+6+4+1+6 pp. (1944).

In this paper the equation of transfer

$$\operatorname{div}\{\Omega\psi(\mathbf{r},\Omega)\}+\psi(\mathbf{r},\Omega)=\frac{1-\alpha}{4\pi}\int\psi(\mathbf{r},\Omega)d\Omega,$$

where Ω is a unit vector and $\alpha<1$ is an assigned constant, is considered in a manner which will be applicable to all geometries. The method consists in expanding $\psi(\mathbf{r},\Omega)$ in terms of spherical harmonic tensors $T_{i_1\cdots i_n}$ defined in the following manner:

$$T_{i_1\cdots i_n}=\frac{(-1)^n}{n!}D_{i_1\cdots i_n}\left(\frac{1}{r}\right)=\frac{(-1)^n}{n!}D_{i_1}D_{i_2}\cdots D_{i_n}\left(\frac{1}{r}\right),$$

where D_i is the operation of covariant differentiation. Explicitly $T_{i_1\cdots i_n}$ has the form

$$\begin{aligned}T_{i_1\cdots i_n}=\frac{1}{n!}\{&1\cdot3\cdot5\cdots(2n-1)\Omega_{i_1}\cdots\Omega_{i_n}\\&-1\cdot3\cdot5\cdots(2n-3)\textstyle\sum\Omega_{i_1}\cdots\Omega_{i_n}g_{i_1i_2}\\&+1\cdot3\cdot5\cdots(2n-5)\textstyle\sum\Omega_{i_1}\cdots\Omega_{i_n}g_{i_1i_2}g_{i_3i_4}-\cdots\},\end{aligned}$$

where the Ω_i's are the covariant components of Ω, g_{ij} is the covariant metric tensor and $\sum$ is to be understood as summation over all the different terms which may be formed by rearranging subscripts. The corresponding contravariant tensors $T^{i_1\cdots i_n}$ are defined analogously in terms of the contravariant components of Ω^i of Ω.

Writing the equation of transfer in the form

$$D_j\{\Omega^j\psi(\mathbf{r},\Omega)\}+\psi(\mathbf{r},\Omega)=\frac{1-\alpha}{4\pi}\psi^{(0)}(\mathbf{r})$$

and multiplying this equation in turn by T^{i_1}, $T^{i_1i_2}$, $T^{i_1i_2i_3}$, etc., we obtain an infinite system of equations for the moments $\psi^{i_1\cdots i_n}=\int\psi(\mathbf{r},\Omega)T^{i_1\cdots i_n}d\Omega$. The method of solution now consists in considering the finite system obtained by putting equal to zero all moments of order higher than a given one.

In the approximation in which all moments of order less than $2n$ are included, the system reduces to the single equation

$$\prod_{i=1}^{n}(\nabla^2-k_i^2)\psi_0(\mathbf{r})=0,$$

where the k_i's are the roots of the characteristic equation

$$1=(1-\alpha)\sum_{j=1}^{n}a_j/(1-k^2\mu_j^2),$$

where the $\pm\mu_j$'s are the zeros of $P_{2n}(\mu)$ and the a_j's are the corresponding Christoffel members. The characteristic roots are shown in this manner to be independent of the geometry.

The solutions for the case of plane, spherical and cylindrical symmetries are considered separately. In the two latter cases the solutions involve Bessel functions of half odd integral and integral orders respectively.

S. Chandrasekhar (Williams Bay, Wis.).

Mark, J. Carson. The spherical harmonic method. II. (Application to problems with plane and spherical symmetry.) National Research Council of Canada. Division of Atomic Energy. Document no. **CRT-338** (N.R.C. **1589**), i+56 pp. (1945).

The equations derived in the first paper [see the preceding review] are considered in detail for the cases of plane and spherical geometries. Tables of solutions and constants are provided. [In the plane case the solutions obtained are equivalent to those of S. Chandrasekhar, Astrophys. J. **100**, 76–86, 117–127 (1944); **103**, 165–192 (1946); **104**, 110–132, 191–202 (1946); these Rev. **6**, 76, 190; **7**, 489; **8**, 59, 178.]

S. Chandrasekhar (Williams Bay, Wis.).

Functional Analysis, Ergodic Theory

Graves, L. M. What is a functional? Amer. Math. Monthly **55**, 467–472 (1948).

Expository article.

Whitney, Hassler. On ideals of differentiable functions. Amer. J. Math. **70**, 635–658 (1948).

Soit A l'anneau topologique des fonctions numériques $f(x)$ m fois continuement différentiables sur l'espace numérique à n dimensions R^n (extension immédiate à une variété différentiable); la topologie est celle de la convergence uniforme sur tout compact de R^n de la fonction f et de ses dérivées d'ordre non supérieur à m. En chaque point x de R^n l'anneau A définit un anneau ponctuel A_x, quotient de A par l'idéal des fonctions f nulles en x ainsi que leurs dérivées d'ordre non supérieur à m. (On peut représenter chaque élément de A_x par un polynome de degré non supérieur à m, et l'image de $f\in A$ par l'application canonique $A\to A_x$ est alors le polynome de Taylor de f limité à l'ordre m.) L'image de tout idéal I de A par l'application $A\to A_x$ est alors un idéal I_x de A_x, idéal ponctuel associé à I en x. L'auteur démontre le théorème: Tout idéal fermé I de A est entièrement caractérisé par l'ensemble de ses idéaux ponctuels I_x, $x\in R^n$. Démonstration basée sur des méthodes déjà développées par l'auteur [Trans. Amer. Math. Soc. **36**, 63–89 (1934)]. Ce théorème joue un rôle important dans l'étude des anneaux normés (A est un anneau normé dans lequel tout idéal fermé sera l'intersection de ses diviseurs primaires). Il était connu pour $m=0$ [anneau des fonctions continues sur un espace compact: Stone, Trans. Amer. Math. Soc. **41**, 375–481 (1937), voir theorème 85, p. 473] et avait été démontré récemment par Šilov pour 1 dimension [$n=1$, m quelconque: Trav. Inst. Math. Stekloff **21** (1947); ces Rev. **9**, 596]. Ces deux cas sont beaucoup plus élémentaires. *L. Schwartz* (Nancy).

Hewitt, Edwin. Rings of real-valued continuous functions. I. Trans. Amer. Math. Soc. **64**, 45–99 (1948).

Ce travail comprend deux parties. Dans la première, l'auteur donne un exposé d'ensemble de la théorie de l'anneau $C^*(X, R)$ des fonctions réelles continues et bornées dans un espace topologique X, et y rattache la compactification bien connue de Stone-Čech pour les espaces complètement réguliers; la plupart des résultats de cette partie sont maintenant classiques. La seconde partie est beaucoup plus originale: l'auteur y fait pour la première fois une étude approfondie de l'anneau $C(X, R)$ de toutes les fonctions réelles (bornées ou non) continues dans un espace complètement régulier X; il munit cet anneau de la topo-

logie dans laquelle les voisinages de 0 sont les ensembles de fonctions telles que $|f(x)| \leqq m(x)$, où $m(x)$ est une fonction de $C(X, R)$ (bornée ou non) telle que $m(x)>0$ pour tout $x \in X$; cette topologie est compatible avec la structure d'anneau de $C(X, R)$. L'auteur caractérise de diverses manières les espaces (qu'il appelle "pseudo-compacts") où toute fonction réelle continue est bornée. Il étudie d'autre part en détail les idéaux maximaux de $C(X, R)$, qui sont fermés et se divisent en deux catégories: les idéaux fixes, formés des fonctions s'annulant en un point $p \in X$, et les autres idéaux maximaux, dits idéaux libres. Si $\mathfrak{M}$ est un idéal maximal libre dans $C(X, R)$, l'auteur montre que le corps quotient $C(X, R)/\mathfrak{M}$ est naturellement muni d'une relation d'ordre qui en fait un corps ordonné, contenant R, mais en général non-archimédien. Ces corps sont toujours identiques à R lorsque X est pseudo-compact, et réciproquement; l'autre cas extrême est celui où pour tout idéal maximal libre $\mathfrak{M}$ de $C(X, R)$, le corps quotient $C(X, R)/\mathfrak{M}$ est non-archimédien; lorsqu'il en est ainsi l'auteur dit que X est un Q-espace. Il donne d'autres caractérisations de ces espaces, qui forment une classe très étendue, comprenant, outre les espaces compacts (bicompacts) les espaces métriques séparables et les espaces localement compacts dénombrables à l'infini; un espace normal n'est pas nécessairement un Q-espace, et inversement, un exemple (dérivé de l'exemple de Niemytzki et van Dantzig) montre qu'un Q-espace peut être non normal.

L'auteur introduit pour tout espace complètement régulier X sa complétion νX pour la structure uniforme la moins fine rendant uniformément continues les fonctions de $C(X, R)$: c'est un Q-espace qui a des propriétés analogues à celles de la compactification de Stone-Čech βX; l'auteur montre d'ailleurs aussi comment l'espace βX peut être obtenu à partir des idéaux maximaux de $C(X, R)$, dont il étudie les relations avec les idéaux maximaux de $C^*(X, R)$. Enfin, il étudie les relations entre deux espaces complètement réguliers X, Y tels que $C(X, R)$ et $C(Y, R)$ soient algébriquement isomorphes; en particulier, si X et Y sont deux espaces métriques séparables, ils sont nécessairement homéomorphes.

Le mémoire contient malheureusement d'assez nombreuses erreurs. Le théorème 13 est trivialement faux, car le produit d'un cercle S_1 et d'un intervalle ouvert est dense dans la sphère S_2, qui n'est pourtant pas homéomorphe au produit de S_1 et d'un intervalle fermé. Le théorème 14 est aussi inexact, comme l'a montré P. Samuel par un contre-exemple [Trans. Amer. Math. Soc. 64, 100–132 (1948); ces Rev. 10, 54]. Il en est de même du théorème 15, et cela déjà dans le produit de R et d'un intervalle compact. La validité du théorème 61, qui dépend du théorème 14, demeure douteuse. Il en est de même en ce qui concerne le théorème 42, car l'auteur y utilise le lemme suivant: la plus petite racine réelle d'un polynome de degré impair à coefficients réels est fonction continue de ces coefficients, ce qui est trivialement faux, comme le montre déjà le polynome $(x^2+a)(x-1)$. Enfin le rapporteur signale que les espaces appelés "espaces de Stone" par l'auteur ont déjà été considérés par N. Bourbaki [C. R. Acad. Sci. Paris 212, 215–218 (1941); ces Rev. 3, 136] qui démontre à leur sujet la première partie du théorème 4. *J. Dieudonné.*

Kaplansky, Irving. Lattices of continuous functions. II. Amer. J. Math. 70, 626–634 (1948).

[Cf. Bull. Amer. Math. Soc. 53, 617–623 (1947); ces Rev. 8, 587.] Soit X un espace compact, $C(X)$ l'ensemble des fonctions continues numériques sur X. L'auteur détermine d'abord les automorphismes de la structure d'ordre de $C(X)$ qui sont aussi des homéomorphismes de $C(X)$ pour la topologie de la convergence uniforme; il donne un exemple d'un automorphisme de structure d'ordre qui n'est pas continu pour cette topologie; toutefois, cela ne peut se produire si X satisfait au premier axiome de dénombrabilité. L'auteur cherche ensuite à déterminer des propriétés d'un "lattice" L qui entraînent l'isomorphie de sa structure d'ordre et d'un $C(X)$. Il obtient un résultat dans cette direction en considérant des "lattices de translation" L où, en outre de la structure d'ordre de "lattice distributif," est définie une loi externe $(\alpha, x) \to x+\alpha$ avec comme domaine d'opérateurs les nombres réels α, et soumise aux axiomes suivants: $x+0=x$; $(x+\alpha)+\beta=x+(\alpha+\beta)$; $\alpha>0$ entraîne $x+\alpha>x$; $x>y$ entraîne $x+\alpha>y+\alpha$; pour tout couple x, y d'éléments de L, il existe α tel que $x+\alpha>y$; et enfin: si $x<y+\alpha$ pour tout $\alpha>0$, alors $x \leqq y$. *J. Dieudonné.*

Turowicz, A. Sur les fonctionnelles continues et multiplicatives. Ann. Soc. Polon. Math. 20 (1947), 135–156 (1948).

Let E be a compact metric space and let $\mathfrak{A}(E)$ be the ring of continuous real-valued functions defined over E. A continuous real-valued functional F defined on $\mathfrak{A}(E)$ is multiplicative if $F(xy)=F(x)F(y)$ for all $x, y \in \mathfrak{A}(E)$. It is shown that there exists a finite or denumerable subset $Z \subset E$ depending on F such that, if $x_1(t)=x_2(t)$ for $t \in Z$, then $F(x_1)=F(x_2)$. The principal theorem gives a representation of the general continuous multiplicative functional over $\mathfrak{A}(E)$ and states that a necessary and sufficient condition that F be continuous and multiplicative is that there exist an ordering $t_1, t_2, \cdots$ of the elements of Z, a sequence $\{\alpha_n\}$ of real numbers, $\alpha_n>0$, $\sum_{n=1}^{\infty}\alpha_n<\infty$, and a sequence $\{\beta_n\}$, where $\beta_n=1$ or 2 for each n, such that

$$F(x)=\prod_{i=1}^{\infty}|x(t_i)|^{\alpha_i}\operatorname{sgn}^{\beta_1}x(t_1)\operatorname{sgn}^{\beta_2}x(t_2)\times\prod_{i=1}^{\infty}[\operatorname{sgn}x(t_{2i+1})\operatorname{sgn}x(t_{2i+2})]^{\beta_{i+2}},$$

where the products are finite in case Z is finite. Each α_n is uniquely determined by F and t_n while the β_n are uniquely determined by F, t_n and the ordering of Z.

R. E. Fullerton (Madison, Wis.).

Katětov, Miroslav. On convex topological linear spaces. Acta Fac. Nat. Univ. Carol., Prague no. 181, 20 pp. (1948).

Section one of this paper is introductory. The first part of section two is devoted to a study of three kinds of infinite direct product for convex topological linear spaces. Concerning these it is proved that if $\{X_\alpha\}$ and $\{Y_\alpha\}$ are families of convex topological linear spaces such that, for each α, X_α and Y_α are dual, then the "Cartesian product" of the X_α is dual to the "lower combinatorial product" of the Y_α and the "upper combinatorial product" of the X_α is dual to the "upper combinatorial product" of the Y_α. The remainder of section two is concerned with "maximal" and "minimal" spaces. A convex topological linear space is maximal (minimal) if every convex topology in the space having the same continuous linear functionals as the given one is stronger (weaker) than or equal to the given one. Typical theorems are as follows. If a dense subspace of a space is maximal (minimal) then the space itself is maximal (minimal). The

upper combinatorial product of maximal (minimal) spaces is again maximal (minimal).

Let X and Y be dual. For each $A \subseteq X$ let A^* be the set of all y in Y with $|xy| \leqq 1$ for all x in A. The author calls sets A and B in X and Y respectively polar if $A = B^*$ and $B = A^*$. In section three various properties of this notion of polarity are discussed. A typical result asserts that if A and B are polar then A is bounded if and only if B is a "convex solid."

Throughout the paper theorems are stated and proved when possible for generalized convex topological linear spaces in which no separation axiom is postulated. As the author acknowledges, a few of his results are known. However, for some of these he offers alternative proofs.

G. W. Mackey (Cambridge, Mass.).

Nikol'skiĭ, V. N. Best approximation and basis in a Fréchet space. Doklady Akad. Nauk SSSR (N.S.) 59, 639–642 (1948). (Russian)

The author starts from a basis $\{u_n\}$ in an F-space [cf. Banach, Théorie des Opérations Linéaires, Warsaw, 1932] and states, with a few indications of proof, that there exists a metric $((u))$, topologically equivalent to the original F-metric, such that for all n the polynomial $\sum_1^n c_k u_k$ making $((u - \sum_1^n c_k u_k))$ a minimum is the nth partial sum of $u = \sum_1^\infty c_k u_k$. Furthermore, he shows that there exists a metric which in addition to this has the following property. The series $\sum_{n+1}^\infty c_k u_k$ making $((u - \sum_{n+1}^\infty c_k u_k))$ a minimum is the nth remainder of $u = \sum_1^\infty c_k u_k$; this metric can be chosen so that $((tu))$ is monotone for any u and $t > 0$. Finally, a necessary and sufficient condition for $\{u_n\}$ to be a basis in an F-space is that (i) it is complete, (ii) it is possible to introduce a metric (u) equivalent to the original F-metric such that for all c_k, n and $m < n$ the polynomial $\sum_1^m a_k u_k$, making $(\sum_1^m a_k u_k - \sum_1^n c_k u_k)$ a minimum, is $\sum_1^m c_k u_k$.

František Wolf (Berkeley, Calif.).

Ryll-Nardzewski, C. Un théorème sur la convergence uniforme dans l'intérieur. Colloquium Math. 1, 145–147 (1948).

A sequence of real-valued functions defined over an open set G in a finite-dimensional Euclidean space is said to converge uniformly in the interior of G if it converges uniformly on any closed subset of G. In the topology defined by uniform convergence in the interior many well-known spaces of functions defined over G are spaces of type (F). In this paper it is shown that a necessary and sufficient condition that such a space M be a Banach space is that there exist a closed subset $F_0 \subset G$ such that every sequence $f_n \in M$ which converges to 0 uniformly on F_0 converges uniformly to 0 in the interior of G.

R. E. Fullerton.

Munroe, M. E. Homomorphisms on Banach spaces. Bull. Amer. Math. Soc. 54, 776–781 (1948).

If E is a Banach space, G a linear subspace, E^* and G^* their respective conjugate spaces, then Krein and Šmulian [Ann. of Math. (2) 41, 556–583 (1940); these Rev. 1, 335] have shown that there is a natural homomorphism T of E^* onto G^* continuous and open in the norm topologies. The author discusses the behaviour of T when E^* and G^* have each one of the usual weak topologies. He proves that T is always continuous and open, and that it is never closed; but that in certain of the weak topologies T transforms restricted classes of closed sets into closed sets.

J. L. B. Cooper (London).

Schatten, Robert. On projections with bound 1. Ann. of Math. (2) 48, 321–325 (1947).

Let f_i and g_i denote elements of the Banach spaces $\mathfrak{A}$ and $\mathfrak{B}$. The greatest crossnorm γ of an expression $\sum f_i \otimes g_i$ is defined as $\inf \sum |f_i'| \cdot |g_i'|$, where the infimum is taken over all equivalent expressions. The central result of this paper is that if $\mathfrak{M}$ is a closed linear manifold in $\mathfrak{A}$ then a necessary and sufficient condition that for every $\mathfrak{B}$ the γ for $\mathfrak{M} \otimes \mathfrak{B}$ coincides with the γ of $\mathfrak{A} \otimes \mathfrak{B}$ where both are defined is that there be a projection of bound 1 on $\mathfrak{M}$ from $\mathfrak{A}$. (A somewhat better result is given for the sufficiency: the γ condition need only hold for a specified $\mathfrak{B}$.) Using a result of Kakutani, one can characterize a Hilbert space by means of this condition.

F. J. Murray.

Michal, Aristotle D. On a non-linear total differential equation in normed linear spaces. Acta Math. 80, 1–21 (1948).

The differential equation in question is

$$\delta y(x; \delta x) = T(y(x), \delta x, y(x)).$$

Here $T(y_1, x, y_2)$ is a function on $B_1 B_2 B_1$ to B_1, where B_1 and B_2 are Banach spaces; it is assumed to be a bounded linear function of each of its arguments. The problem is to find a function $y(x)$ on B_2 to B_1 such that $y(x)$ and its Fréchet differential $\delta y(x; \delta x)$ satisfy the differential equation and the initial condition $y(0) = y_0$ (a given element of B_1). Let $T^n(y_1, x, y_2) = T(y_1, x, T^{n-1}(y_1, x, y_2))$, $n = 2, 3, \cdots$. The author assumes that there exists a positive function $M(y)$ defined on B_1 such that $\|T^n(y, x, y)\| \leqq \|y\| (M(y))^n \|x\|^n / n!$, $n = 1, 2, \cdots$, and he further assumes that T satisfies certain associativity conditions (among them the condition of complete integrability) which are certainly fulfilled if $T\{T(y_1, x_2, y_2), x_1, y_3\} = T\{y_1, x_2, T(y_2, x_1, y_3)\}$. It is then proved that the differential system has a solution $y(x) = y_0 + \sum_{n=1}^\infty T^n(y_0, x, y_0)$, and that this solution is unique within the class of entire functions on B_2 to B_1 (analytic in the sense of R. S. Martin and A. D. Michal). As an illustration take B_1 as the space of continuous functions $y(t, s)$ from the square $a \leqq s \leqq b$, $a \leqq t \leqq b$ to a complete normed linear ring N with a unit I, and B_2 as the space of continuous functions $x(s)$ on $a \leqq s \leqq b$ to N. Take $T(y_1, x, y_2) = z(t, s) = \int_s^t y_1(t, r) x(r) y_2(r, s) dr$. If in particular N is the ring of real $n \times n$ matrices, and $y_0(t, r) = I$, the solution $y(x|t, s)$ of the differential equation is the matrizant functional of the matrix $x(r)$. Thus in particular the matrizant is characterized by an equation satisfied by its Fréchet differential. Other Fréchet differential systems are considered, some of which give rise to theorems about the solution of an ordinary linear differential system as a functional of the matrix coefficients.

A. E. Taylor.

Krein, M. G. On some questions related to the ideas of Lyapunov in the theory of stability. Uspehi Matem. Nauk (N.S.) 3, no. 3(25), 166–169 (1948). (Russian)

The author considers differential equations of the form $dx/dt = F(x, t)$ where x is an element of a Banach space and $F(x, t)$ is an operator defined over the space with values in the space. The relation between the nonlinear equation and its linear approximation is considered, and some theorems are stated which represent generalizations of results due to Perron, Persidsky, and Malkin for the case of systems of ordinary differential equations. No proofs are given.

R. Bellman (Stanford University, Calif.).

Lorch, E. R. On certain implications which characterize Hilbert space. Ann. of Math. (2) 49, 523–532 (1948).

The author proves a number of necessary and sufficient conditions in order that a (real) Banach space $\mathfrak{B}$ be a Hilbert space. The following two conditions of this kind are known: (A) $f, g \in \mathfrak{B}$ implies $|f+g|^2+|f-g|^2=2(|f|^2+|g|^2)$ [cf. P. Jordan and J. von Neumann, same Ann. (2) 36, 719–723 (1935)]; (B) $f, g \in \mathfrak{B}$ and $|f|=|g|$ imply $|\alpha f+\beta g|=|\alpha g+\beta f|$ for all real α, β [cf. F. A. Ficken, same Ann. (2) 45, 362–366 (1944); these Rev. 5, 270]. It is shown first that (B) may be replaced by the weaker condition (B′) involving an arbitrarily fixed pair of real numbers α, β such that $\alpha \neq 0$, $\beta \neq 0$, $\alpha \neq \beta$. In the special case $\alpha=1$, $\beta=-\frac{1}{2}$, this means geometrically that if two sides of a triangle are equal then the corresponding medians are also equal, or, equivalently, that if the two medians are equal then the sides are also. The necessity of the following condition may be derived from (A), its sufficiency from (B′): (C) there is a (nontrivial) functional relation $F(x, y, u, v)=0$ such that $f, g \in \mathfrak{B}$ implies $F(|f|, |g|, |f+g|, |f-g|)=0$. This is, of course, a greatly weakened form of (A). Geometrically speaking, it means that if the three sides of a triangle are respectively equal to the three sides of another, then the corresponding medians are also equal. The following condition involves an inequality rather than an equality: (D) $f, g \in \mathfrak{B}$ and $|f|=|g|$ imply $|\alpha f+\alpha^{-1}g| \geqq |f+g|$ for all real $\alpha \neq 0$. Finally, there is the condition: (E) $f, g, h \in \mathfrak{B}$ and $f+g+h=0$ imply $|f-g|^2+|g-h|^2+|h-f|^2=6(|f|^2+|g|^2+|h|^2)$.

B. de Sz. Nagy (Szeged).

Julia, Gaston. Les systèmes orthonormaux complets et leurs généralisations dans l'espace hilbertien. C. R. Acad. Sci. Paris 227, 317–320 (1948).

The author extends the results of two previous notes [same C. R. 226, 1485–1487 (1948); 227, 168–170 (1948); these Rev. 9, 516; 10, 48] from finite dimensional unitary spaces to Hilbert spaces. *P. R. Halmos* (Chicago, Ill.).

Julia, Gaston. Sur les systèmes complets de l'espace hilbertien. C. R. Acad. Sci. Paris 227, 413–416 (1948).

The sum of the squares of the lengths of the projections of a complete orthonormal set in Hilbert space onto a linear manifold V is equal to the dimension of V. The author discusses a generalization and a converse of this theorem and applies his results to obtain and generalize the results of Hilding on the completeness of certain sets [Ark. Mat. Astr. Fys. 32B, no. 7 (1946); these Rev. 8, 151].

P. R. Halmos (Chicago, Ill.).

Coddington, Earl A. Note on the spectral representation of a bounded normal matrix. Bull. Amer. Math. Soc. 54, 736–739 (1948).

The spectral representation of a bounded normal (infinite) matrix is obtained, in a natural way, from the integral representation of completely monotone double sequences.

B. de Sz. Nagy (Szeged).

Mihlin, S. G. On the convergence of Galerkin's method. Doklady Akad. Nauk SSSR (N.S.) 61, 197–199 (1948). (Russian)

Let H be a separable Hilbert space and A a linear operator defined on a linear manifold dense therein. Let it be required to solve the equation (1) $Au-f=0$, where f is a given element of H. Let $\{\phi_n\}$ be a complete basis in H. Then Galerkin's method, as explained by the author, consists in finding a sequence of approximate solutions $\{u_n\}$, where u_n is a linear combination of $\phi_1, \cdots, \phi_n$ such that Au_n-f is orthogonal to $\phi_1, \cdots, \phi_n$. The principal theorem announced is that if $Au=A_0u+\lambda Ku$, where A_0 is positive definite and self-adjoint, and if λ is a numerical parameter, then the sequence $\{u_n\}$ converges in the following two cases: (I) the operator A has an inverse which is defined everywhere in H; (II) the operator $T=A_0^{-1}K$ is completely continuous in a certain subspace H. (Certain auxiliary conditions are omitted here.) Further in case II the Galerkin method leads to a convergent process in connection with the characteristic value problem. A number of applications are given in which the author states that he has shown the hypotheses are fulfilled. There are neither proofs nor references, except a single citation in the case of one application. *H. B. Curry* (State College, Pa.).

Barankin, Edward W. Bounds on characteristic values. Bull. Amer. Math. Soc. 54, 728–735 (1948).

Soit $A=(a_{st})$, $s, t=1, 2, \cdots$, une matrice infinie, non nulle, de nombres complexes. Si $p>0$, on dira que la matrice A est absolument sommable (p) si l'on a $\sum_{s,t}|a_{st}|^p<\infty$. En posant $R_s^{(p)}=\sum_t|a_{st}|^p$, $T_t^{(p)}=\sum_s|a_{st}|^p$, l'auteur démontre, sur les valeurs propres λ de A, les théorèmes suivants. (I) Si A est absolument sommable (1), on a $|\lambda|^2 \leqq \max_{(s)} R_s^{(1)}T_s^{(1)}$, $|\lambda| \leqq \max_{(s)} R_s^{(1)}$, $|\lambda| \leqq \max_{(t)} T_t^{(1)}$. (II) Si A est absolument sommable (p) avec $p \leqq 1$, on a $|\lambda|^2 \leqq \max_{(s)} R_s^{(q)}T_s^{(2-q)}$ avec $p \leqq q \leqq 2-p$; plus généralement, pour une arbitraire matrice (q_{st}), $s, t=1, 2, \cdots$, avec $p \leqq q_{st} \leqq 2-p$ (sauf au plus pour un nombre fini d'éléments), on a $|\lambda|^2 \leqq \max_{(s)} (\sum_t|a_{st}|^{q_{st}} \cdot \sum_r|a_{rs}|^{2-q_{rs}})$. Il établit aussi des résultats analogues pour les valeurs propres (relatives à des fonctions propres $f(s)$ mesurables et bornée sur G_s) d'une équation intégrale $\lambda f(s)=\int_{G_t}K(s, t)f(t)dt$ avec le noyau $K(s, t)$ sommable sur $G_s \times G_t$. En posant $R(s)=\int_{G_t}|K(s, t)|dt$, $T(s)=\int_{G_s}|K(t, s)|dt$, on a

$$|\lambda|^2 \leqq \sup_{(s \in G_s)} R(s)T(s), \quad |\lambda| \leqq \sup_{(s \in G_s)} R(s), \quad |\lambda| \leqq \sup_{(s \in G_s)} T(s).$$

A. Ghizzetti (Rome).

Riesz, Frederick. On a recent generalisation of G. D. Birkhoff's ergodic theorem. Acta Univ. Szeged. Sect. Sci. Math. 11, 193–200 (1948).

The fundamental hypothesis in Birkhoff's ergodic theorem is that the transformation T (which is not necessarily one-to-one) be measure preserving; that is, the set of points $T^{-1}(Te)$ should have the same measure as the set Te. Dunford and Miller [Trans. Amer. Math. Soc. 60, 538–549 (1946); these Rev. 8, 280] have extended the Birkhoff result by requiring merely that $n^{-1}\sum_1^n|T^{-k}e| \leqq K|e|$ for some fixed constant k; here $|e|$ denotes the measure of the set e. The author proves that this generalization can be obtained by a modification of the proof he has given of Birkhoff's original theorem [Comment. Math. Helv. 17, 221–239 (1945); these Rev. 7, 255]. *E. R. Lorch.*

Calculus of Variations

Damköhler, Wilhelm. Über die absoluten Minimanten des regulären semidefiniten Variationsproblems. Math. Z. 51, 150–160 (1948).

When a variation problem of minimizing an integral taken along a path γ,

$$I(\gamma)=\int F(x, \dot{x})dt,$$

is regular semidefinite but not definite, there may exist one or more "Carathéodory continua" $\mathfrak{C}$ with the property that through each point of $\mathfrak{C}$ there passes a "null extremal" E lying in $\mathfrak{C}$ for which $I(E)=0$. For the case when there is only one such continuum $\mathfrak{C}$ let $f(x)=\sup I(\gamma)$ for rectifiable curves γ beginning at x and ending on $\mathfrak{C}$; $g(x)=\sup I(\gamma)$ for rectifiable curves γ beginning on $\mathfrak{C}$ and ending at x; $\mu(x_1, x_2)=\sup I(\gamma)$ for rectifiable curves γ joining x_1 to x_2. For points x_1 and x_2 not on $\mathfrak{C}$ there exists an extremal E_1', leading from x_1 and spiraling around $\mathfrak{C}$, and an extremal E_2'' spiraling around $\mathfrak{C}$ and leading to x_2. The author's principal theorem states that when (1) $\mu(x_1, x_2)<f(x_1)+g(x_2)$, every minimizing curve for I between x_1 and x_2 is rectifiable. Conversely, if I has a rectifiable minimizing curve between x_1 and x_2 then either (1) holds, or else the following statement holds: for each point x_1' on E_1' between x_1 and $\mathfrak{C}$ and each point x_2' on E_2'' between $\mathfrak{C}$ and x_2 the integral I has no rectifiable minimizing curve between x_1' and x_2'; and for each point x_1'' on the backward extension of E_1' and each point x_2'' on the forward extension of E_2'', the integral I has a rectifiable minimizing curve between x_1'' and x_2''. An extension is given to the case of many continua $\mathfrak{C}$. The author's reasoning seems unclear at a number of points. In particular, for the case when space is two-dimensional, $\mathfrak{C}$ is a closed Jordan curve, and the points x_1 and x_2 are on the same side of $\mathfrak{C}$, it is not clear why the extremals E_1' and E_2'' must intersect [Satz 3]. In the proof of Satz 5, it is not clear why the null extremals E_n lying on $\mathfrak{C}_n$ should have lengths with a positive lower bound.

L. M. Graves (Chicago, Ill.).

Faedo, Sandro. Il calcolo delle variazioni per gli integrali su un intervallo infinito. Pont. Acad. Sci. Comment. **8**, 319–421 (1944).

This paper is a comprehensive exposition of results concerning calculus of variations integrals $I_\infty(C)$ extended over infinite intervals. A number of these results have already been published in various places. Extensions of existence theorems of Nagumo and McShane to $I_\infty(C)$ are given in chapter II, with the restriction that the integrand f is nowhere less than a function $\psi(x)$ which is integrable over the infinite interval. (Absolute convergence of integrals over the infinite intervals is not required.) In chapter III, the first variation of $I_\infty(C)$ is considered, and equations for extremals and extremaloids are derived under additional restrictive assumptions. It is noted that the first variation need not be zero along an extremal, and in chapter V an example is included to show that the first variation need not be zero even along a minimizing curve. In chapter IV it is shown under suitable conditions that an extremal which makes $I_\infty(C)$ finite and makes the first variation vanish is uniquely determined by the values assigned at a fixed finite abscissa. In chapter V it is shown by examples that when the condition that the first variation vanish is omitted in the preceding, the remaining conditions sometimes uniquely determine an extremal, and sometimes do not. Under restrictive hypotheses, transversality conditions at ∞ can be derived which can be used to restore uniqueness. Chapter VI secures existence theorems for the case of quadratic integrands by means of comparison with another suitably chosen integrand for which the corresponding integral is already known to have a minimum furnished by an extremaloid. Examples of comparison integrands are given. The author refrains from considering minimum problems in which boundary conditions at infinity are prescribed, since there are many such problems in which no minimizing curve exists. The same situation holds for integrals over finite intervals which fail to be everywhere regular. Some modifications in the hypotheses and no doubt also in the method of proof will be needed to secure existence theorems when boundary conditions are prescribed at infinity as well as at the finite end of the interval. The paper contains numerous misprints, some of which are confusing. *L. M. Graves.*

Lawson, J. W. A sufficiency theorem for the Plateau problem. Trans. Amer. Math. Soc. **64**, 192–204 (1948).

For the classical problem of minimizing the area of a surface through a single boundary, Schwarz stated necessary conditions and sufficient conditions for a surface to yield a minimum to the corresponding multiple integral problem; two of these conditions are stated in terms of the spherical representation [see T. Radó, On the Problem of Plateau, Ergebnisse der Math., v. 2, no. 2, Springer, Berlin, 1933], while one condition is stated in terms of the tangent plane [see L. Bianchi, Vorlesungen über Differentialgeometrie, 2d ed., Teubner, Leipzig-Berlin, 1910]. It is this latter result that forms the model for the present paper. The author obtains analogues of the preceding results for $(n-1)$-dimensional hypersurfaces M_n in Euclidean n-space E_n, with rather general boundary, with a straightforward development of the corresponding multiple integral problem, culminating in a proof of the following principal theorem. Let M_n be a minimal hypersurface of class C'', which does not intersect itself, and let the boundary of M_n be L. If there is a point in E_n through which there passes no tangent hyperplane to M_n, then there is a neighborhood F of M_n, in E_n, such that M_n is minimizing among all (admissible) hypersurfaces in F having boundary L. This main theorem is used to discuss minimizing properties of the catenoid in E_3.

M. O. Reade (Ann Arbor, Mich.).

Lévy, Paul. Le problème de Plateau. Mathematica, Timişoara **23**, 1–45 (1948).

The author studies situations where the Plateau problem has several solutions. The starting point is an intuitively evident continuity consideration, and the mathematical problem consists of proving the implications of this intuitive consideration. The methods used are related to those developed by Morse and Tompkins. The author states that he developed his work, in the main, without knowing about similar work in the United States. He also states that certain parts of his argument are in the heuristic stage as yet.

T. Radó (Columbus, Ohio).

Theory of Probability

Lévy, Paul. Étude d'une nouvelle classe de permutations. C. R. Acad. Sci. Paris **227**, 578–579 (1948).

From a deck of n cards the top card is put on the table, the next is placed at the bottom of the pack, and the procedure is repeated alternating the two possibilities until the entire deck is on the table. The author is interested in the permutation thus defined. He studies its order and various other properties. *W. Feller* (Ithaca, N. Y.).

Lévy, Paul. The arithmetical character of the Wishart distribution. Proc. Cambridge Philos. Soc. **44**, 295–297 (1948).

It is shown that the Wishart distribution is indecomposable as defined in the author's "Théorie de l'Addition des Variables Aléatoires" [Gauthier-Villars, Paris, 1937].

L. A. Aroian (New York, N. Y.).

Lovera, Giuseppe. Questioni statistiche sulla distribuzione temporale degli eventi casuali. Ricerca Sci. **17**, 2042–2044 (1947).

Consider random events obeying the Poisson law and let $p_k(D, T, m)$ be the conditional probability that an interval of length D contains exactly k events which are separated from the preceding event by an interval of length exceeding T, when it is known that the interval contains exactly m events. Using the method of Feller [Phys. Rev. **57**, 906–908 (1940); these Rev. **1**, 344] the author finds the mean and variance of the distribution $p_k(D, T, m)$, and also of the distribution $p_k(D, T) = \sum_m p_k(D, T, m) \cdot \pi_m(D)$, where π_m is the Poisson expression for the probability of exactly m events in an interval of length D. *W. Feller.*

Klemens, P. G. Resolution of devices actuated by random events. Philos. Mag. (7) **39**, 656–660 (1948).

The author supposes that a counter operates in such a way that each recording is followed by a resolving time τ_1 and each event not recorded is followed by a resolving time τ_2; here τ_1 and τ_2 are constants. The incoming events obey the Poisson law. The mean number of recordings is calculated. *W. Feller* (Ithaca, N. Y.).

Domb, C. Some probability distributions connected with recording apparatus. Proc. Cambridge Philos. Soc. **44**, 335–341 (1948).

Let events occur in accordance with the Poisson law, and let a recording apparatus have a constant resolving time τ so that no event can be recorded during an interval τ following a registration. Let $x_n(t)$ and $y_n(t)$ be the probabilities that in the interval $(0, t)$ exactly n registrations occur and that at time t the apparatus is alive or dead, respectively. The author obtains two difference-differential equations for $x_n(t)$ and $y_n(t)$ which he solves using Laplace transforms. He then treats the analogous problem for the more general case where τ is a random variable. In a note added in proof the author notes the similarity of his method to that of L. Kosten [Physica **10**, 749–756 (1943); these Rev. **6**, 161]. [Cf. also Feller, Studies and Essays Presented to R. Courant, Interscience Publishers, New York, 1948, pp. 105–115; these Rev. **9**, 294.] *W. Feller.*

Stansfield, R. G. Statistical theory of d.f. fixing. J. Inst. Elec. Engrs. Part III A. **94**, 762–770 (1947).

Using the normal error theory the author solves various probability problems concerning the most likely position of an object when a set of bearings is given. *W. Feller.*

Gordon, A. N. The restricted problem of the random walk. Philos. Mag. (7) **39**, 572–575 (1948).

Elementary remarks concerning the unrestricted symmetric random walk in 1, 2, 3 dimensions. The familiar formula for the probability of a specified displacement is derived by means of generating functions. *W. Feller.*

Marks, Eli S. A lower bound for the expected travel among m random points. Ann. Math. Statistics **19**, 419–422 (1948).

It is proved that if m points are chosen independently and at random (probability proportional to area) in a plane region of area A, and if L is the length of the shortest path going through every point, then $E(L) \geqq (\frac{1}{2}A/m)^{\frac{1}{2}}(m-1)$.

J. L. Doob (Urbana, Ill.).

Kanellos, S. G. Statistical test of an observation on logarithmic tables by a method of Pearson. Bull. Soc. Math. Grèce **23**, 127–131 (1948). (Greek. English summary)

A statistical confirmation of the well-known fact that the second digit in logarithmic tables does not correspond to Bernoulli trials. *W. Feller* (Ithaca, N. Y.).

Bruges, W. E. The curve of error in which the maximum error is defined. Philos. Mag. (7) **39**, 394–399 (1948).

The author starts from the thesis, attributed to Whittaker and Robinson, that the normal approximation to the binomial distribution "is not capable of exact proof." His summary reads: "It is shown that the curve of error deduced from the binomial theorem has a limit and that if, for example, the probable error is known, then the maximum error should not exceed 3.48 ··· times this error."

W. Feller (Ithaca, N. Y.).

Gabriel, Fritz. Durch Höchstfehler begrenzte Fehlerverteilungen. Z. Angew. Math. Mech. **28**, 244–247 (1948).

The author is perturbed by the fact that the normal distribution has infinite tails and recommends a suggestion of Jordan [Handbuch der Vermessungskunde, v. 1, 4th ed., Metzler, Stuttgart, 1895] the use of the density function

$$\phi(x) = \tfrac{1}{2} \cdot \frac{3 \cdot 5 \cdot 7 \cdot \cdots \cdot (2N+1)}{2 \cdot 4 \cdot 6 \cdot \cdots \cdot (2N)} \cdot M^{-1}(1 - x^2/M^2)^N$$

restricted to $|x| < M$. Some properties of $\phi(x)$ are discussed.

W. Feller (Ithaca, N. Y.).

May, Kenneth. Probabilities of certain election results. Amer. Math. Monthly **55**, 203–209 (1948).

This paper is concerned with the probability that a candidate will win an election with a minority of the votes when there is an indirect voting system such as that of an electoral college. Of course it is necessary to consider a comparatively simple electoral system and in particular the author assumes that there are two parties, that the number of voters is the same for each electoral district, that the districts are independent and that each is subject to a flat distribution with respect to the outcome of the vote. The probability varies with the number of districts and the number of voters in each district but is remarkably stable with respect to changes in these numbers and is of the order of magnitude of $\frac{1}{3}$. It is difficult to tell how these results would be affected by more realistic assumptions but it can be said that dependence would tend to decrease the probability whereas more centrally concentrated distributions would tend to increase it. *A. H. Copeland* (Ann Arbor, Mich.).

Lukacs, Eugene. On the mathematical theory of risk. J. Inst. Actuaries Students' Soc. **8**, 20–37 (1948).

Expository paper.

Borel, Émile. Sur les développements unitaires normaux. Ann. Soc. Polon. Math. **21**, 74–79 (1948).

Proofs of results announced in C. R. Acad. Sci. Paris **225**, 773 (1947); cf. also P. Lévy, ibid. 918–919 (1947) [these Rev. **9**, 292]. *W. Feller* (Ithaca, N. Y.).

Chung, Kai Lai. On the maximum partial sums of sequences of independent random variables. Trans. Amer. Math. Soc. **64**, 205–233 (1948).

A thorough study of the behavior of $S_n^* = \max_{\nu \leq n} |S_\nu|$, where S_ν is the sum of ν independent random variables. Assumptions (usual notations): $s_n \to \infty$, $\max_{\nu \leq n} \gamma_\nu \sigma_\nu^{-3} = O(s_n^{1-\theta})$, θ arbitrarily small but fixed. Upper and lower bounds for the distribution of S_n^* are obtained sharpening Erdös and Kac's methods [Bull. Amer. Math. Soc. **52**, 292–302 (1946); these Rev. **7**, 459] by use of the Berry-Esseen theorem. Estimates of remainders are deduced. Almost-certain precise upper and lower bounds for S_n^*, which are the exact counterparts of Feller's for S_n [same Trans. **54**, 373–402 (1943); these Rev. **5**, 125], are then established by similar arguments. *M. Loève* (Berkeley, Calif.).

Gnedenko, B. V. On a local limit theorem of the theory of probability. Uspehi Matem. Nauk (N.S.) **3**, no. 3(25), 187–194 (1948). (Russian)

Let $\xi_1, \xi_2, \cdots$ be mutually independent random variables with a common distribution. It is supposed that the ξ_j's assume only integral values and have finite second moments. It is proved that under these hypotheses

$$\lim_{n \to \infty} [n^{\frac{1}{2}} \Pr \{\xi_1 + \cdots + \xi_n = k\} - w_{nk}] = 0$$

uniformly in k if and only if 1 is the highest common factor of the values assumed with positive probability by the ξ_j's. Here w_{nk} is the value (Gaussian density) suggested by the central limit theorem. Previously van Kampen and Wintner [Amer. J. Math. **61**, 965–973 (1939); these Rev. **1**, 63] obtained weaker results in that in their work $k = cn^{\frac{1}{2}} + o(n^{\frac{1}{2}})$, but treated multidimensional random variables, with no number-theoretic restrictions on the integral values assumed by them. [Cf. the following review.] *J. L. Doob.*

Meĭzler, D. G., Parasyuk, O. S., and Rvačeva, E. L. A multidimensional local limit theorem of the theory of probability. Doklady Akad. Nauk SSSR (N.S.) **60**, 1127–1128 (1948). (Russian)

The authors generalize Gnedenko's result [cf. the preceding review] to multidimensional random variables. The proof is omitted. *J. L. Doob* (Urbana, Ill.).

Silber, Jack. Multiple sampling for variables. Ann. Math. Statistics **19**, 246–256 (1948).

The paper deals with the following problem, arising in sequential analysis as well as in random walk and risk theory. Let $x_1, \cdots, x_n$ be independent random variables with identical frequency functions $f(x)$, and let $s_n = x_1 + \cdots + x_n$ $(n = 1, 2, \cdots)$. Given two limits a, b $(a < b)$, it is desired to determine the probability that s_n sooner or later passes above the limit b (or below a) without having earlier left the interval (a, b); furthermore, one wishes to determine the mean time n at which s_n leaves this interval. [Cf., for a related problem, Khintchine, Asymptotische Gesetze der Wahrscheinlichkeitsrechnung, Ergebnisse der Math., v. 2, no. 4, Springer, Berlin, 1933, p. 32.] The probabilities and the mean mentioned are found to equal $\int_b^\infty Y(x)dx$, $\int_{-\infty}^a Y(x)dx$, $1 + \int_a^b Y(x)dx$, respectively, where $Y(x)$ satisfies the integral equation $Y(x) = f(x) + \int_a^b f(x-s)\,Y(s)ds$; $Y(x)$ may, for $x < a$ and $x > b$, be interpreted as the frequency function of the first value s_n outside (a, b). Explicit solutions of the problem are obtained for the case when $f(x)$ is rectangular, by reducing the integral equation to a set of differential-difference equations. *G. Elfving.*

Moran, P. A. P. The statistical distribution of the length of a rubber molecule. Proc. Cambridge Philos. Soc. **44**, 342–344 (1948).

A chain consists of n links of fixed length such that successive links are at a fixed angle to each other with uniform probability density for all permissible directions. The author shows that the length of the chain is asymptotically normally distributed. The proof is based on S. Bernstein's form of the central limit theorem for dependent variables. *W. Feller* (Ithaca, N. Y.).

Moran, P. A. P. A class of complex Markoff chains. Quart. J. Math., Oxford Ser. **19**, 140–149 (1948).

The author considers a sequence of random variables $x_1, x_2, \cdots$ with the following properties: $x_n = \pm 1$; if $n > k$ the probability that $x_n = 1$ for given $x_1, \cdots, x_{n-1}$ depends only on the values of the last k of the x_j's and is $\frac{1}{2} - \sum_{j=1}^{k} a_j x_{n-j}$, where $\sum_{j=1}^{k} |a_j| < \frac{1}{2}$. The x_n process is then a complex Markov chain. It is proved that with the proper initial probabilities the x_n process is stationary and [using S. Bernstein's results] that the central limit theorem is applicable to the sequence $x_1, x_2, \cdots$ and to the sequence $x_1 y_1, x_2 y_2, \cdots$, where the y_n process is defined like the x_n process, not necessarily with the same a_j's, and is independent of it. [These results can be derived from the known theory of ordinary Markov chains, using the fact that the x_n process is equivalent to an ordinary Markov chain with 2^k instead of 2 states, and that the (x_n, y_n) process is equivalent to an ordinary Markov chain with 4^k states.] The covariance function of the stationary x_n process is calculated explicitly, as well as the limiting variances in the central limit theorem. *J. L. Doob* (Urbana, Ill.).

Everett, C. J., and Ulam, S. Multiplicative systems. I. Proc. Nat. Acad. Sci. U. S. A. **34**, 403–405 (1948).

What the authors call a multiplicative system is usually referred to as a stationary branching process. The authors consider the case where particles are of different kinds. Each particle has well-defined probabilities of transmuting into specified groups of particles of the various types, and the particles are mutually independent in the statistical sense. Usually one considers only certain random variables (like population size) associated with the process. The authors show how the boundary process itself can be described in terms of graphs and they discuss the appropriate measure set up in a sample space. *W. Feller.*

Blanc-Lapierre, André. Remarques sur l'analyse harmonique des fonctions aléatoires. Revue Sci. **85**, 1027–1040 (1947).

The author gives a detailed exposition with examples of results centering around the harmonic analysis of (not necessarily stationary) stochastic processes. The results, based on the work of Loève [cf., for example, Revue Sci. **84**, 195–206 (1946); these Rev. **8**, 281], have been announced without details of proofs in earlier papers [cf. particularly C. R. Acad. Sci. Paris **225**, 982–984, 1264–1266 (1947); these Rev. **9**, 293]. *J. L. Doob* (Urbana, Ill.).

Kendall, David G. On some modes of population growth leading to R. A. Fisher's logarithmic series distribution. Biometrika 35, 6–15 (1948).

The author discusses a "birth-and-death process" for which the Kolmogorov differential equations assume the form

$$P_n'(t) = -\{n(a+b)+c\}P_n(t)+\{(n-1)a+c\}P_{n-1}(t) + (n+1)bP_{n+1}(t),$$

where $P_n(t)$ is the probability of a population size n. The case $c=0$ corresponds to mortality and fertility proportional to the actual population size. The c-term accounts for an increase by immigration. The generating function of $P_n(t)$ is obtained and it is shown that for small c one obtains approximations to R. A. Fisher's "logarithmic series distribution" which has found several applications in biology.

W. Feller (Ithaca, N. Y.).

Woodward, P. M. A statistical theory of cascade multiplication. Proc. Cambridge Philos. Soc. 44, 404–412 (1948).

The discrete stationary branching process where each "parent" has the probability p_n of producing exactly n direct descendants has been studied by F. Galton, R. A. Fisher, Steffensen, Kolmogorov, and others. It was rediscovered independently by many authors. The author uses the familiar technique of iterating the generating function to obtain moments of the distribution of the population size. [Cf., for example, A. J. Lotka, Théorie Analytique des Associations Biologiques. II, Actual. Sci. Ind., no. 780, Hermann, Paris, 1939, in particular, pp. 123 ff.]

W. Feller (Ithaca, N. Y.).

Lotka, Alfred J. Application of recurrent series in renewal theory. Ann. Math. Statistics 19, 190–206 (1948).

In the discrete bounded case the integral equation of renewal theory reduces to the recurrence formula

$$B(k) = \sum_1^N c_\nu B(k-\nu),$$

where the c_ν and $B(t)$ for $t<0$ are given. The author discusses the method of representing the solution as a linear combination of terms x_τ^{-k}, where the x_τ are roots of the algebraic equation $\sum c_\nu x^\nu = 1$. Numerical illustrations are given.

W. Feller (Ithaca, N. Y.).

Wiener, Norbert. Time, communication, and the nervous system. Ann. New York Acad. Sci. 50, 197–220 (1948).

Cf. the author's "Cybernetics" [Wiley, New York, 1948; these Rev. 9, 598].

Shannon, C. E. A mathematical theory of communication. Bell System Tech. J. 27, 379–423, 623–656 (1948).

The author considers an information source which can produce any one of a finite number of symbols. These are fed into a channel for transmission, each symbol having a positive time duration in the channel. The problem becomes statistical with the assumption that if x_n is the nth symbol produced by the source the x_n process is a stationary stochastic process. (Further hypotheses of Markov type are also made in the applications.) The concept of the entropy $H(U)$ of a random variable U which can assume only a finite number of values, with probabilities $p_1, p_2, \cdots$, is fundamental: $H(U) = -\sum_i p_i \log p_i$. The entropy is a measure of the uncertainty in the value of U, and has various suggestive properties indicating this; for example $H(U)$ is greater than or equal to the entropy of U conditioned by the knowledge of a second random variable. If the x_n's above are mutually independent the entropy of the information source is defined as $H(x_n)$, and is a measure of the amount of information available in the source for transmission. A slightly modified definition of source entropy is made if the x_n's are not mutually independent. The capacity of a channel is defined as $C = \lim_{T\to\infty} \log N(T)/T$, where $N(T)$ is the maximum number of sequences of symbols that can be transmitted in time T, and it is shown that for given H and C the symbols can be encoded in such a way that $C/H-\epsilon$ symbols per second can be transmitted over the channel if $\epsilon>0$ but not if $\epsilon<0$. Many examples are given and the effect of noise on the transmission is treated in some detail. For example, if $H \leqq C$ above, there exists a coding system such that the output of the source can be transmitted over the channel with an arbitrary small error frequency. [For the early beginnings of this theory cf. Nyquist, same J. 3, 324–346 (1924); Hartley, ibid. 7, 535–563 (1928).]

In the second part of the paper the author treats absolutely continuous distributions. The entropy of a distribution in one or more dimensions, with density $p(x)$, is defined as $-\int p(x) \log p(x)\, dx$ and various qualitative properties of entropy are discussed. For example, the entropy of a distribution in one dimension, with finite dispersion σ^2, is a maximum (relative to a not very clearly specified set of unbounded distributions) for a normal distribution. The entropy per degree of freedom of a stationary and ergodic process with random variables $x_1, x_2, \cdots$ is defined as $H' = \lim_{n\to\infty} H_n/n$, where H_n is the entropy of the distribution of $x_1, \cdots, x_n$, and it is stated that if $p(x_1, \cdots, x_n)$ is the density of the latter distribution, then $n^{-1} \log p$ converges in probability to H'. The entropy decrease caused by the action of a linear filter on a stationary ergodic process is evaluated. The rate R of transmission of information over a channel is defined and the channel capacity is then defined as the maximum of R for an arbitrarily varying input. For example the capacity of a channel of band 0 to W cps perturbed by white noise of power N, for average transmission power P, is $W \log(P+N)/P$ and this can be interpreted to mean that proper coding will allow the transmission of this many binary digits of information per second with arbitrarily small error frequency. Finally the rate of generating information relative to a certain degree of fidelity is defined and discussed. The discussion is suggestive throughout, rather than mathematical, and it is not always clear that the author's mathematical intentions are honorable. The point of view is that stressed by Wiener in his NDRC report [soon to be published as a book] "The Interpolation, Extrapolation, and Smoothing of Stationary Time Series," in which communication is considered as a statistical problem, specifically in its mathematical formulation as the study of stationary stochastic processes, and of the results of various operations performed on them.

J. L. Doob (Urbana, Ill.).

Bennett, W. R. Spectra of quantized signals. Bell System Tech. J. 27, 446–472 (1948).

The author presents a detailed study of the harmonic analysis of signals quantized in time of occurrence and in magnitude. Involved in the latter is the following mathematical problem. Let $\{x(t)\}$ be the random variables of a stationary Gaussian stochastic process with $E\{x(t)\}=0$. Let $g(\xi)=\xi-mE$ for $\frac{1}{2}(2m-1)E \leqq \xi < \frac{1}{2}(2m+1)E$. Then if $x(t)$ represents a signal, for example noise, $\epsilon(t)=g[x(t)]$ is the error in quantizing the signal in steps of magnitude E

by a staircase transducer. The $\epsilon(t)$ process is stationary, and the author finds its covariance function and thereby its spectral distribution in terms of the covariance function of the $x(t)$ process. *J. L. Doob* (Urbana, Ill.).

Campbell, N. R., and Francis, V. J. Random fluctuations in a cathode ray oscillograph. Philos. Mag. (7) **37**, 289–310 (1946).

The theory of noise is developed from the so-called "shot effect representation" [see Rice, Bell System Tech. J. **23**, 282–332 (1944); H. Hurwitz, Jr., and Kac, Ann. Math. Statistics **15**, 173–181 (1944); Rivlin, Philos. Mag. (7) **36**, 688–693 (1945); these Rev. **6**, 89; **8**, 39]. The results are known [Rice, loc. cit.] but the methods of derivation are often new. The treatment is not quite rigorous.

M. Kac (Ithaca, N. Y.).

Mathematical Statistics

Kerawala, S. M. On bounds of skewness and kurtosis. Bull. Calcutta Math. Soc. **40**, 41–44 (1948).

Let $x_1, \cdots, x_n$ be nonidentical real numbers,

$$\alpha_r = \{\sum(x_i-\bar{x})^r/n\}/\{\sum(x_i-\bar{x})^2/n\}^{r/2},$$

$r=1, 2, \cdots$; p any positive integer. Then $-M_{2p+1}$ and M_{2p+1} are the lower and upper bounds for α_{2p+1}, where $M_{2p+1}=\{(n-1)^{2p}-1\}/n(n-1)^{p-1}$. The upper bound for α_{2p+2} is $\{(n-1)^{2p+1}+1\}/n(n-1)^p$, the lower bound being 1 if n is even and $\{(n+1)^{2p+1}+(n-1)^{2p+1}\}/2n(n^2-1)^p$ if n is odd. The author quotes known results for the case $p=1$; it is not clear to whom these special cases should be attributed. *T. E. Harris* (Santa Monica, Calif.).

Gengerelli, J. A. A simplified method for approximating multiple regression coefficients. Psychometrika **13**, 135–146 (1948).

Roy, S. N. A note on critical angles between two flats in hyperspace with certain statistical applications. Sankhyā **8**, 177–194 (1947).

Stange, K. Über die Verteilungsdichte der Mess- oder Beobachtungsfehler eines dreidimensionalen Punktraumes. Z. Angew. Math. Mech. **28**, 235–243 (1948). (German. Russian summary)

Using tensor terminology the author essentially starts with the assumption that three stochastic variables are linear forms in a set of independent stochastic variables each obeying a normal distribution law, derives the three-dimensional normal distribution function in a standard manner and notes various well-known properties.

C. C. Craig (Ann Arbor, Mich.).

Schützenberger, Marcel-Paul. Valeurs caractéristiques du coefficient de corrélation par rang de Kendall dans le cas général. C. R. Acad. Sci. Paris **226**, 2122–2123 (1948).

Using the "deviations from independence" introduced by him [same C. R. **225**, 277–278 (1947); these Rev. **9**, 96] the author gives expressions for the mean and variance of Kendall's coefficient of rank correlation τ [Biometrika **30**, 81–93 (1938)] between two independent sets of n drawings from the values of a stochastic variable obeying an arbitrary distribution law. If, in particular, one of the sets is fixed in its numerical order, the author obtains an upper bound of the variance of τ. *C. C. Craig.*

Hoeffding, Wassily. A class of statistics with asymptotically normal distribution. Ann. Math. Statistics **19**, 293–325 (1948).

Suppose $X_\nu=(X_\nu^{(1)}, \cdots, X_\nu^{(r)})$, $\nu=1, \cdots, n$, are independent random vectors having the same cumulative distribution function $F(x)$. Let $\Phi(x_1, \cdots, x_m)$ be a function of $m(\leqq n)$ vectors $x_\nu=(x_\nu^{(1)}, \cdots, x_\nu^{(s)})$, and set

$$U=[(n-m)!/n!]\sum\Phi(X_{\alpha_1}, \cdots, X_{\alpha_m}),$$

where $\alpha_1, \cdots, \alpha_m$ are distinct integers among $1, 2, \cdots, n$ and $\sum$ denotes summation over all permutations of $\alpha_1, \cdots, \alpha_n$. The statistic U is an unbiased estimate of the quantity $\theta(F)=E(\Phi(X_1, \cdots, X_m))$, called a regular functional of $F(x)$. The author shows that

$$\lim_{n\to\infty} \Pr[n^{\frac{1}{2}}(U-\theta(F))<y]=(2\pi)^{-\frac{1}{2}}\int_{-\infty}^{y} e^{-\frac{1}{2}t^2}dt$$

assuming only that $E[\Phi(X_1, \cdots, X_m)]^2$ exists. He also considers the joint distribution of several statistics of the type to which U belongs, the various statistics differing by taking different choices of integers from $1, 2, \cdots, n$. If the statistics are $U^{(1)}, \cdots, U^{(s)}$, and the corresponding population parameters are $\theta^{(1)}(F), \cdots, \theta^{(s)}(F)$, the author shows that the quantities $n^{\frac{1}{2}}[U-\theta^{(i)}(F)]$ are asymptotically jointly normally distributed for large n. The case in which the random vectors $X_1, \cdots, X_n$ are independent but have different distributions is considered with the conclusion that under rather straightforward assumptions the U statistics are asymptotically distributed. A modified U statistic defined by $U'=U+n^{-\frac{1}{2}}b_n$ is considered, where $\lim_{n\to\infty} E(b_n^2)=0$, and it is shown that its limiting distribution properties for large n are identical with those of U.

The limiting distribution theory of the U and U' statistics is applied to the sampling theory of Fisher's k-statistics, Gini's mean difference, the rank correlation coefficient, Kendall's partial difference sign correlation coefficient, etc.

S. S. Wilks (Princeton, N. J.).

Robbins, Herbert. The distribution of Student's t when the population means are unequal. Ann. Math. Statistics **19**, 406–410 (1948).

Let $x_1, \cdots, x_N$ be independent normal variates with the same variance σ^2 and means $\mu_1, \cdots, \mu_N$, respectively. Define $\bar{\mu}=\sum\mu_i/N$, $\beta^2=\sum(\mu_i-\bar{\mu})^2/N$, $\alpha=N\bar{\mu}^2/2\sigma^2$, $\lambda=N\beta^2/2\sigma^2$, $n=N-1$, $\bar{x}=\sum x_i/N$, $s^2=\sum(x_i-\bar{x})^2/n$, $t=N^{\frac{1}{2}}\bar{x}/s$; all summations extend from 1 to N. The author finds the distribution of t in the general case $\bar{\mu}\neq 0$ and the special case $\alpha=\bar{\mu}=0$. If $x_1, \cdots, x_{N_1+N_2}$ are independent normal variates with the same variance σ^2 and means $\mu_1, \cdots, \mu_{N_1+N_2}$, respectively, and with similar definitions for $\bar{x}_1$, $\bar{x}_2$, s_1^2, s_2^2, s^2, t, $\bar{\mu}_1$, $\bar{\mu}_2$, β_1^2, β_2^2, n, the distribution of t is the same as before. The results are of importance in situations involving mixtures of populations or if the mean is subject to a secular trend. *L. A. Aroian* (New York, N. Y.).

Das, A. C. A note on the D^2-statistic when the variances and co-variances are known. Sankhyā **8**, 372–374 (1948).

This note shows that two different distributions of the difference of the sample means from two p-variate normal populations are identical. These two distributions had been derived by R. C. Bose [Sankhyā **2**, 143–154, 379–384 (1936)], who had decided that the first one was not correct.

R. L. Anderson (Raleigh, N. C.).

Bose, P. K. Parametric relations in multivariate distributions. Sankhyā 8, 167–171 (1947).

The incomplete integral of the probability function of the D^2 statistic in multivariate analysis is reduced to a form suitable for numerical calculations. *L. A. Aroian.*

Nanda, D. N. Limiting distribution of a root of a determinantal equation. Ann. Math. Statistics 19, 340–350 (1948).

Let $x=\|x_{ij}\|$, $x^*=\|x^*_{ij}\|$ be two p-variate sample matrices with n_1 and n_2 degrees of freedom and $S=\|xx'\|/n_1$, $S^*=\|x^*x^{*\prime}\|/n_2$ the covariance matrices which under the null hypothesis are independent estimates of the same population covariance matrix; then the joint distribution of the roots of the determinantal equation $|A-\theta(A+B)|=0$, where $A=n_1S$, $B=n_2S^*$, was obtained by Hsu [Ann. Eugenics 9, 250–258 (1939); these Rev. 1, 248]; here $l=\min(p, n_1)$, $\mu=|p-n_1|+1$, and $\nu=n_2-p+1$. If $m=\mu/2-1$, $n=\nu/2-1$, the author determines the limiting distribution of the largest root as $n\to\infty$ for $l=2$, 3, 4 and 5; the limiting distribution for the smallest root $l=2$, 3, and for the intermediate root in case $l=3$. The methods employed are based on previous results of the author [Ann. Math. Statistics 19, 47–57 (1947); these Rev. 9, 453]. *L. A. Aroian.*

Guttman, Louis. A distribution-free confidence interval for the mean. Ann. Math. Statistics 19, 410–413 (1948).

Consider a random sample, of mean m and variance s^2, from a population of mean μ and variance σ^2. It is shown by using the Chebyshev inequality that

$$\Pr\{(m-\mu)^2\leqq s^2/(N-1)+\lambda\sigma^2[2/N(N-1)]^{\frac{1}{2}}\}\geqq 1-\lambda^{-2}.$$

A method for obtaining further inequalities of this kind is outlined. *M. Loève* (Berkeley, Calif.).

Noether, Gottfried E. On confidence limits for quantiles. Ann. Math. Statistics 19, 416–419 (1948).

Confidence limits that are order statistics are obtained for the p-quantile $(0<p<1)$ of an arbitrary cumulative distribution function by means of confidence limits for the binomial parameter. *D. F. Votaw, Jr.*

Yates, F. Systematic sampling. Philos. Trans. Roy. Soc. London. Ser. A. 241, 345–377 (1948).

This paper discusses various results dealing with one-dimensional systematic sampling. No fully reliable estimate of the sampling error can be obtained from the observations themselves, except that, under certain circumstances, the sum of sets of terms taken alternately positive and negative provides a reasonably satisfactory estimate (which is usually an overestimate). The method of "partial systematic samples," based on short sections of completely enumerated sequences, is proposed for estimating the systematic sampling error. A number of more specific populations are investigated from the point of view of the results of systematic sampling. They are: (1) a general function which takes only the values 0 and 1 (sampling for attributes); (2) the normal distribution function; (3) a first order stochastic difference equation. Five numerical sequences are analyzed and it is claimed that the numerical results confirm the theoretical investigations. The author's summary describes the dangers inherent in applying systematic sampling to material about which nothing is known. It is suggested that applications of systematic sampling be preceded by preliminary sampling investigations. Provision should also be made for supplementary sampling, if need be.

J. Wolfowitz (New York, N. Y.).

Bhattacharyya, A. On some analogues of the amount of information and their use in statistical estimation (concluded). Sankhyā 8, 315–328 (1948).

[For parts I and II, see Sankhyā 8, 1–14 (1946); 201–218 (1947); these Rev. 8, 524; 9, 365.] Applications are given of the results of parts I and II. Let $f(x_1, \cdots, x_n; \theta_1, \cdots, \theta_l)$ be a given n-dimensional, l-parameter density function. One of the results: let $t_1, t_2, \cdots$, be parameter-free functions of the x_r which can be expressed as linear functions, with coefficients depending only on the θ_r, of quantities $(1/f)\partial^{i_1+\cdots+i_l}f/\partial\theta_1^{i_1}\cdots\theta_l^{i_l}$; then any convergent power series in the t_r is the best (minimum variance) estimate of its own expectation. Special applications of this and other results are made to distributions for which sets of sufficient statistics exist, and to the Cauchy, multivariate normal, gamma, beta and multinomial distributions. Modifications of the general results are given for the case where functional constraints exist among the parameters. *T. E. Harris.*

Wald, Abraham. Estimation of a parameter when the number of unknown parameters increases indefinitely with the number of observations. Ann. Math. Statistics 19, 220–227 (1948).

Let $X_1, X_2, \cdots$ be an infinite sequence of random variables, such that for each n the variables $X_1, \cdots, X_n$ admit a continuous joint probability density $p_n(x_1, \cdots, x_n|\theta, \xi_1, \cdots, \xi_n)$, where $\theta, \xi_1, \cdots, \xi_n$ are unknown parameters, all of which are restricted to finite intervals. Then $t_n(X_1, \cdots, X_n)$ is said to be a uniformly consistent estimate of θ if $\Pr(|t_n-\theta|<\delta)\to 1$ as $n\to\infty$, for any $\delta>0$, uniformly in θ and the ξ's. Necessary and sufficient conditions for the existence of a uniformly consistent estimate are given, using results from an earlier paper of the author [Ann. of Math. (2) 46, 265–280 (1945); these Rev. 7, 21]. An analogue of R. A. Fisher's information function is defined for the present case, and a sufficient condition for the nonexistence of a uniformly consistent estimate is given in terms of the information function. In the particular case when the X_i are independent, the total information contained in the first n observations is equal to the sum of the amounts of information contained in each observation separately.

H. Cramér (Stockholm).

Hoel, Paul G. On the uniqueness of similar regions. Ann. Math. Statistics 19, 66–71 (1948).

The author proves that a certain set of conditions on a parametric probability distribution function f is sufficient for: (i) the uniqueness of similar regions constructed by Neyman's "sufficient"-statistic method [Philos. Trans. Roy. Soc. London. Ser. A. 236, 333–380 (1937)]; (ii) the equivalence of the "sufficient"-statistic method and another method of Neyman [same Ann. 12, 46–76 (1941); these Rev. 3, 9] for constructing similar regions, which requires that f satisfy certain partial differential equations. *D. F. Votaw, Jr.*

Ghosh, M. N. On the problem of similar regions. Sankhyā 8, 329–338 (1948).

It has been shown by Neyman that, if a sufficient set of statistics exists, then one can construct similar regions. The regions constructed in this way have the property that their

conditional size on each surface on which the sufficient statistics are all constant is independent of the level at which they are held constant. Such regions are said by the author to have a structure of detailed similarity.

The author shows, by means of an example, that there exist similar regions which do not possess the structure of detailed similarity. He then gives a set of sufficient conditions that every similar region have the desired structure. They are that

$$p(x_1, \cdots, x_n) = (1/F)\left(\exp \sum_{i=1}^{k} g_i T_i\right) G,$$

where F, g_i are functions of the parameters alone, and G, T_i are functions of $x_1, \cdots, x_n$ alone, and these functions possess certain regularity properties. The conditions are essentially the same as those given by Hoel [see the preceding review] and by Lehman and Scheffé [Proc. Nat. Acad. Sci. U. S. A. 33, 382–386 (1947); these Rev. 9, 365]. *K. J. Arrow.*

Albert, G. E. Correction to "A note on the fundamental identity of sequential analysis." Ann. Math. Statistics 19, 426–427 (1948).

The paper appeared in the same Ann. 18, 593–596 (1947); these Rev. 9, 296.

Nandi, H. K. Use of well-known statistics in sequential analysis. Sankhyā 8, 339–344 (1948).

The possibility of using ordinary estimators of parameters in sequential testing is examined. For sufficient estimators there is no loss in efficiency relative to the Wald probability ratio test, but for parameters without sufficient estimators the Wald test is more efficient. It is shown that if T_n is an estimator of θ with density $f_n(T_n; \theta)$ then the sequential test defined by continued sampling when $A < f_n(T_n, \theta_1)/f_n(T_n, \theta_0) < B$ will terminate under quite general conditions. *A. M. Mood* (Santa Monica, Calif.).

Girault, Maurice. Sur la notion de facteur commun en analyse factorielle générale. C. R. Acad. Sci. Paris 227, 499–500 (1948).

The random variables x and y are defined by $x = F(a, b)$, $y = G(a, c)$, where a, b and c are independent random variables and F and G are regular functions. In the usual model for factor analysis F and G are linear functions with nonzero coefficients of a, and then x and y are dependent. In this paper it is shown that, if $F = ab$, $G = \{2 - (4 + a^2 - 4ac)^{\frac{1}{2}}\}/a$, x and y are independent when a, b, and c have independent uniform distributions on the interval $(-1, 1)$. The author gives a necessary and sufficient condition on F and G for x and y to be independent. *T. W. Anderson.*

TOPOLOGY

Descartes, Blanche. Network-colourings. Math. Gaz. 32, 67–69 (1948).

This paper deals with graphs of degree 3 which cannot be factorized into three graphs of degree 1. [Cf. W. W. R. Ball, Mathematical Recreations and Essays, 11th ed., Macmillan, London, 1939; New York, 1947, pp. 224–227; these Rev. 8, 440.] The simplest instance is the Petersen graph, which has ten nodes. Some other instances are trivial modifications of this one. The author finds a nontrivial modification having 90 nodes and conjectures that every such graph is in some sense a complication of the Petersen graph. If this matter could be clarified, it might follow that every such graph is nonplanar, and then the four-color theorem would be established. [Cf. the following review.]

H. S. M. Coxeter (New York, N. Y.).

Blanuša, Danilo. Le problème des quatre couleurs. Hrvatsko Prirodoslovno Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. 1, 31–42 (1946). (Croatian. French summary)

In the note reviewed above B. Descartes suggested that the most hopeful approach to the four-color problem might be to classify graphs of degree three that cannot be factorized into three graphs of degree one, and challenged the reader to find such a graph that is not merely a derivative of the Petersen graph. The author has answered [or rather, anticipated] this challenge with a remarkable graph having 18 nodes and 27 branches. *H. S. M. Coxeter.*

Fáry, István. On straight line representation of planar graphs. Acta Univ. Szeged. Sect. Sci. Math. 11, 229–233 (1948).

The author shows that if a finite graph G can be represented in the plane at all, then it can be represented in the plane with straight segments as edges. He assumes some theorems in point-set topology, e.g., that in a representation of a graph in the plane the edges meeting at any given vertex have a definite cyclic order. *W. T. Tutte.*

Hall, D. W., and Lewis, D. C. Coloring six-rings. Trans. Amer. Math. Soc. 64, 184–191 (1948).

In the study of chromatic polynomials of the n-ring, one encounters a system of simultaneous equations whose coefficients are polynomials. These have been fully discussed for four- and five-rings where there are respectively 3 and 6 equations. As a step towards the higher cases, in this paper the system of 15 equations for the six-ring are solved. The methods used suggest a possible inductive theory, and the results have been fully checked by geometric methods. *P. Franklin* (Cambridge, Mass.).

Muracchini, Luigi. Sulla specie degli angoli dei poliedri generalizzati. Boll. Un. Mat. Ital. (3) 3, 32–34 (1948).

A face-angle of a polyhedron is said to be of type (f, v) if it belongs to an f-gonal face and to a v-hedral vertex. It was proved by Lebesgue [J. Math. Pures Appl. (9) 19, 27–43 (1940); these Rev. 1, 316] that every polyhedron of genus 0 has either an angle of type $(3, v)$ for some $v < 6$ or one of type $(f, 3)$ for some $f < 6$. The author proves that every polyhedron of genus greater than 1 has an angle of type (4, 5) or (5, 4) or (5, 5) or else of type (f, v) where either $f > 5$ or $v > 5$. He finds that both these results remain valid for polyhedra of genus 1, provided we set aside the "regular" polyhedra in which all the angles are of the same type: (3, 6) or (6, 3) or (4, 4). *H. S. M. Coxeter.*

Katětov, Miroslav. Remarque sur les espaces topologiques dénombrables. Ann. Soc. Polon. Math. 21, 120–122 (1948).

Let X be absolutely closed, i.e., let it be impossible to adjoin any new nonisolated points to X. If X is regular, then it is (bi)compact [Alexandroff and Urysohn, Verh. Akad. Wetensch. Amsterdam, Afd. Natuurk. Sect. 1. 14, no. 1 (1929)]. If X is semi-regular, it may not be compact [Stone, Trans. Amer. Math. Soc. 41, 375–481 (1937)]. The present paper shows that if X is countable, absolutely

closed, and if X is hereditarily semi-regular, i.e., if all its subspaces are semi-regular, then X is compact. An example is given to show that hereditary semi-regularity does not imply regularity for countable spaces. *R. Arens.*

Ramanathan, A. A characterization of maximal-Hausdorff spaces. J. Indian Math. Soc. (N.S.) 11, 73–80 (1947).

It is proved that the maximal-Hausdorff spaces (i.e., those whose continuous one-to-one images are necessarily homeomorphic images) coincide with the semi-regular H-closed spaces. Moreover, a Hausdorff space which is merely maximal-Hausdorff at some point is already H-closed. The first of these results was also proved by M. Katětov [Časopis Pěst. Mat. Fys. 69, 36–49 (1940), prop. 1.6; these Rev. 1, 317], in a paper which was not known to the author. *R. Arens* (Los Angeles, Calif.).

Stone, M. H. On the compactification of topological spaces. Ann. Soc. Polon. Math. 21, 153–160 (1948).

The principal facts regarding the compactification $\beta(X)$ of a completely regular X were established by the author [Trans. Amer. Math. Soc. 41, 375–481 (1937)] within the framework of a general theory of imbedding. A treatment, somewhat more geometrical and thus more desirable, from an expository point of view, was given later, but independently, by E. Čech [Ann. of Math. (2) 38, 823–844 (1937)]. The proofs here presented of the facts regarding $\beta(X)$ exploit the simplifications possible in the original proofs by emphasizing the geometrical features of Čech's methods. *R. Arens* (Los Angeles, Calif.).

Vedenisov, N. B. Bicompact spaces. Uspehi Matem. Nauk (N.S.) 3, no. 4(26), 67–79 (1948). (Russian)

This is an expository paper. The contents include (1) the various definitions of bicompactness, and proofs of their equivalence in Hausdorff spaces, (2) examples of non-metrizable bicompact spaces, (3) the proof that a topological product of an arbitrary number of bicompact spaces is bicompact, (4) characterizations of closed, of open, and finally of arbitrary subsets of a bicompact space, (5) the bicompactification due to E. Čech [Ann. of Math. (2) 38, 823–844 (1937)]. There is a bibliography. *L. Zippin.*

Balanzat, Manuel. Sur la formation des espaces à écart régulier et symétrique. Revue Sci. 86, 34 (1948).

It is pointed out that, if Ω is the first uncountable ordinal, then 2^{Ω}, lexicographically ordered, is a nonmetrizable example of a space écartizable in the sense of Fréchet [Portugaliae Math. 5, 121–131 (1946); these Rev. 8, 48]. *R. Arens* (Los Angeles, Calif.).

Novák, J. Remark on $\mathfrak{M}$-completeness. Ann. Soc. Polon. Math. 21, 123–124 (1948).

Let P be a class and M a family of metrics for P. Let P^* be obtained by adding just enough ideal elements to P so that sequences which are Cauchy-convergent in all the metrics in M have limits in P^*. The author gives an example to show that P^{**} may properly include P^*. *R. Arens.*

Kelly, Paul J. On isometries of product sets. Bull. Amer. Math. Soc. 54, 723–727 (1948).

Let A and B be arbitrary metric spaces. The author constructs a metric in the product space $A \times B$ in the following way. Let C' be a curve lying in the first quadrant of the Euclidean plane which intersects the axes of X and Y at points different from the origin. Suppose that the curve C, made up of C' and the segments of the coordinate axes which C' intercepts, is (1) a simple convex closed curve, (2) has the property that $x_1 \leqq x_2$ and $y_1 \leqq y_2$ imply that $OP_1/OP_1' \leqq OP_2/OP_2'$, where $P_i = (x_i, y_i)$ and P_i' is the point where OP_i intersects the curve C $(i = 1, 2)$. Such a curve is called a modified Minkowski gauge; if C is symmetric about the line $y = x$, it is called a symmetric gauge. For $P = (x, y)$, let $\|P\| = OP/OP'$. If ρ is the metric in A and σ is the metric in B, then the metric γ in $A \times B$ is defined by the formula $\gamma((a_1, b_1), (a_2, b_2)) = \|(\rho(a_1, a_2), \sigma(b_1, b_2))\|$. The main theorem proved is that if A and B are bounded sets on the real line such that both A and B contain pairs of points whose distances are the diameters of A and B, and if $A \times A$ and $B \times B$ are metrized by the same symmetric modified Minkowski gauge, then an isometry between $A \times A$ and $B \times B$ implies an isometry between A and B. Analogous results are stated for products of different spaces and products involving more than two factor spaces. *E. Hewitt* (Seattle, Wash.).

Gomes, Alfredo Pereira. Sur la fonction diamètre. C. R. Acad. Sci. Paris 226, 2112–2113 (1948).

The author introduces a function, defined axiomatically on a lattice, which is a generalization of the function "diameter" on the lattice of subsets of a metric space. He shows how this yields another approach to uniform structures. *R. Arens* (Los Angeles, Calif.).

Gomes, Alfredo Pereira. Topologie induite par un pseudo-diamètre. C. R. Acad. Sci. Paris 227, 107–109 (1948).

In terms of the concepts of the paper reviewed above, conditions are obtained under which an abstract diameter defined in a lattice induces a closure operation satisfying various of the familiar axioms. *R. Arens.*

Császár, Ákos. An elementary proof of a theorem of H. E. Vaughan. Norske Vid. Selsk. Forh., Trondhjem 20, no. 1, 1–3 (1947).

H. E. Vaughan [Bull. Amer. Math. Soc. 43, 532–535 (1937)] proved that a locally compact separable metric space can be remetrized so that bounded sets are compact. Here this theorem is given a short proof not based on the Urysohn-Alexandroff theory of metrization. *R. Arens.*

Dieudonné, Jean. On topological groups of homeomorphisms. Amer. J. Math. 70, 659–680 (1948).

The author studies groups of homeomorphisms of a space E with a uniform structure in the sense of Weil, and obtains results generalizing previous work of his own [Anais Acad. Brasil. Ci. 18, 287–289 (1946); these Rev. 8, 525], of G. Birkhoff [Ann. of Math. (2) 35, 861–875 (1934)], of R. Arens [same J. 68, 593–610 (1946); these Rev. 8, 479], and of the reviewer [Ann. of Math. (2) 47, 496–502 (1946); these Rev. 8, 165]. Typical results are as follows. Let H be the group of all homeomorphisms of E under the compact-open topology; let G be a subgroup of H. Then (1) if G is equicontinuous, it is a topological group; (2) if E is locally compact and connected, H is a topological group and is complete for its two-sided uniform structure; (3) if G has a symmetrical neighborhood of the identity whose closure in H is compact, then G is a topological group and the set of limit points of Cauchy filters on G (for either right or left uniform structure of G) forms a locally compact topological group $G' \subset H$ in which G is dense; (4) if E is locally compact, connected, and complete, and G has a symmetrical equicontinuous

neighborhood of the identity, then G satisfies the assumption in (3). *S. B. Myers* (Ann Arbor, Mich.).

Scorza Dragoni, Giuseppe. A proposito di un teorema di Terasaka. Pont. Acad. Sci. Acta **10**, 321–337 (1946).

Let t be a topological fixed-point-free automorphism of the plane. A closed region U is free if it does not intersect tU, critical if U and tU have common boundary points but fail to intersect otherwise. Let Q be a critical square region. The union of the successive images of Q form a connected band (possibly with holes). A theorem of Terasaka [Jap. J. Math. **7**, 61–69 (1930)] asserts that either the band generated by Q admits on a given side a band which is adjacent to it (in a specified sense) and is also generated by a critical square, or else the given side contains a free or critical region bounded by a pair of perpendicular rays. The author proves the existence of certain fundamental sets relative to a critical square Q. Assuming that the band generated by Q admits an adjacent band, a fundamental set relative to Q appears roughly to be a free or critical region which overlaps Q and stretches well across the adjacent band. The exact definition is such that, if S is fundamental relative to Q, it is fundamental relative to any critical square whose vertices are sufficiently near those of Q. *P. A. Smith.*

Hu, Sze-Tsen. Mappings of a normal space into an absolute neighborhood retract. Trans. Amer. Math. Soc. **64**, 336–358 (1948).

Hopf's theorem on the classification of mappings of an n-complex into an n-sphere and the associated extension theorems have been generalized by the reviewer [Ann. of Math. (2) **41**, 231–251 (1940); **42**, 459–468 (1941); Lectures on Topology, University of Michigan Press, Ann Arbor, Mich., 1941, pp. 57–99; these Rev. **1**, 222; **3**, 141, 132] to maps of complexes K into arbitrary spaces Y provided that suitable cohomology and homotopy groups of K and Y vanish. By a systematic and efficient use of the Čech cohomology theory these theorems are now established for maps $X \to Y$ where X is normal and Y is an absolute neighborhood retract. Sample theorem: If X is compact, $\dim X \leqq m$, Y is an absolute neighborhood retract, if the homotopy groups $\pi_r(Y)$ vanish for $r < n$ and the Čech cohomology groups $H^r(X, \pi_r(Y))$, $H^{r+1}(X, \pi_r(Y))$ vanish for $n < r \leqq m$ then the homotopy classes of maps $f: X \to Y$ are in a one-to-one correspondence with the elements of the group $H^n(X, \pi_n(Y))$. It is stated that the compactness condition may be dropped if one uses cohomology groups based on suitable infinite coverings as in the work of Dowker [Amer. J. Math. **69**, 200–242 (1947); these Rev. **8**, 594]. *S. Eilenberg* (New York, N. Y.).

Fox, Ralph H. On the imbedding of polyhedra in 3-space. Ann. of Math. (2) **49**, 462–470 (1948).

Unter einer Elementarfigur des 3-dimensionalen sphärischen Raumes R wird eine polyedrale Teilmenge von R verstanden, welche die abgeschlossene Hülle einer Gebietskomponente des Komplementes eines Systems zueinander fremder geschlossener Flächen in R (also eine berandete 3-dimensionale Mannigfaltigkeit) ist; jedes zusammenhängende Polyeder in R ist Deformationsretrakt einer abgeschlossenen Umgebung, welche eine Elementarfigur ist. Der Hauptsatz der vorliegenden Arbeit besagt, dass jede Elementarfigur durch Ausbohren einer Umgebung eines endlichen Streckenkomplexes aus R gewonnen werden kann. Jedes zusammenhängende Polyeder ist also vom Homotopietypus des Aussenraumes eines endlichen Streckenkomplexes in R. Als Anwendung ergeben sich einfache Bedingungen für die Einbettbarkeit eines Polyeders in R; insbesondere wird unter Benützung eines Satzes von Reidemeister [Monatsh. Math. Phys. **43**, 20–28 (1936)] gezeigt: Ist die Fundamentalgruppe eines Polyeders in R Abelsch, so ist sie ein Modul vom Range $\leqq 2$. *B. Eckmann.*

Whyburn, G. T. On n-arc connectedness. Trans. Amer. Math. Soc. **63**, 452–456 (1948).

A simple elementary proof is provided for the well-known theorem [L. Zippin, Ann. of Math. (2) **34**, 95–113 (1933)] that, if M is a locally connected, locally compact, connected metric space and a and b are points of M such that no set of fewer than n points separates a and b in M, then M contains n arcs with endpoints a and b, which intersect in pairs only in a and b. The proof extends to arbitrary n the inductive type of reasoning used by the author in his paper on the cyclic connectivity theorem [Bull. Amer. Math. Soc. **37**, 429–433 (1931)]. *R. L. Wilder* (Ann Arbor, Mich.).

GEOMETRY

Sperner, Emanuel. Die Ordnungsfunktionen einer Geometrie. Arch. Math. **1**, 9–12 (1948).

The author introduces the concept of an order-function for the purpose of making geometric order relations amenable to algebraic formulation and treatment. Consider a set of two kinds of elements, "points" (denoted by α) and "hyperplanes" (denoted by h), and let $h(\alpha)$ be a function defined for each hyperplane h and point α which assumes only the three values $0, 1, -1$. In terms of the "order-function" one defines (1) incidence of hyperplane and point if and only if $h(\alpha) = 0$, (2) hyperplane h is between points α and β if and only if $h(\alpha)h(\beta) = -1$, and (3) hyperplanes h, k separate points α, β if and only if $h(\alpha)k(\alpha)h(\beta)k(\beta) = -1$. Some applications are sketched. *L. M. Blumenthal.*

Blumenthal, Leonard M. Methods and problems of distance geometry. Memorias de Matemática del Instituto "Jorge Juan," no. 5, 45 pp. (1948).

This paper reports on results in abstract distance geometry mainly due to Menger and the author; most of them are already available in the literature. In addition, it contains (a) the proof given by Aronszajn [Erg. Math. Kolloquium. **6**, 45–46 (1935)] for the segment connectedness of a complete convex metric space; (b) a new simple proof under weaker hypotheses for the imbedding of a semimetric space S in an n-dimensional Euclidean space E_n; (c) a sketch of new investigations on systems of linear inequalities $\sum_{i=1}^{n+1} p_i(t)x_i \geqq 0$, where the $p_i(t)$, $i = 1, \cdots, n+1$, are real functions defined on an arbitrary set T. A summary of (c) follows. If C denotes the cone of the vectors λp, where $\lambda \geqq 0$, $p = (p_1(t), \cdots, p_{n+1}(t))$, in E_{n+1} (coefficient set), C^* the (closed) convex closure of C, the set $\Sigma(C)$ of the solutions is the supplementary cone of C^*. The latter cone does not reduce to the null vector, i.e., the system has at least one nontrivial solution, if and only if C lies on one side of a hyperplane in E_{n+1}. Criteria derived from recent results of C. V. Robinson [Amer. J. Math. **64**, 260–272 (1942); these Rev. **3**, 300] are given, which involve finite subsets of T. [Systems of linear inequalities of the type studied in (c)

have been investigated by the reviewer, Rev. Sci. (Rev. Rose Illus.) 77, 657–658 (1939); these Rev. 1, 263.]
C. Y. Pauc (Cape Town).

da Silva Paulo, José. Equivalence of polyhedra. Gaz. Mat., Lisboa 9, no. 36, 4–6 (1948). (Portuguese)
A new simple proof of Dehn's theorem that the sums of the dihedral angles of two polyhedra in E^3 which are "zerlegungsgleich" differ by a multiple of π.
H. Busemann (Los Angeles, Calif.).

Bagemihl, F. On indecomposable polyhedra. Amer. Math. Monthly 55, 411–413 (1948).
An indecomposable polyhedron is one which cannot be dissected into tetrahedra without the introduction of new vertices. A construction is given for simple indecomposable polyhedra of n vertices ($n \geqq 6$) such that every face is a triangle, every triangle of edges is a face, the join of any two vertices is outside of the polyhedron unless it is an edge of the polyhedron, and every tetrahedron of vertices has a region exterior to the polyhedron. *M. Goldberg.*

Botez, Mihail Şt. Sur une nouvelle géométrie côtée. Revista Ştiinţifică "V. Adamachi" 34, 62–65 (1948).
The author proposes a modified cartographic projection. The elevation of a point above the horizontal plane is given by means of an auxiliary view. This leads to a special linear mapping of a familiar type rather than to a new method of projection. *E. Lukacs* (China Lake, Calif.).

Emch, Arnold. Neue durch stereographische Projektion erhaltene Eigenschaften der Flächen zweiter Ordnung mit Nabelpunkten. Monatsh. Math. 52, 189 (1948).
The author proves the following two theorems on the stereographic projection of a quadric surface F_2. (1) The stereographic projection of any conic on F_2 from a real umbilical point is a circle. (2) Let U_1 and U_2 be two real umbilical points on the same diameter and project a conic C_2 on F_2 from these umbilical points on a plane π intersecting F_2 in a circle e. If e_1 and e_2 are the projections of C_2 on π from U_1 and U_2, respectively, then e_1 and e_2 are inverse with respect to e. *E. Lukacs* (China Lake, Calif.).

Sevdić, Milenko. Le triangle et quadrilatère associés dans la géométrie de Lobačevski. Hrvatsko Prirodoslovno Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. 2, 11–17 (1947). (Croatian. French summary)
Bolyai [Appendix Scientiam Spatii Absolute Veram Exhibens, new ed., Budapest, 1902, § 34] proposed the following construction for a line through a given point A parallel to a given line ED. Draw AE perpendicular to ED, then AB perpendicular to AE, then DB perpendicular to AB, and take C on BD in such a position that $AC=ED$; then AC is parallel to ED, as desired. The author proves that the acute angle D of the trirectangle $ABDE$ is equal to the angle of parallelism for the distance BC.
H. S. M. Coxeter (New York, N. Y.).

Gambier, Bertrand. Courbes planes de classe p dont tous les systèmes de tangentes concourantes ont les mêmes directions de p-sectrices. Bull. Sci. Math. (2) 71, 232–246 (1 plate) (1947).
Let (δ, D) denote one of the angles, determined both in magnitude and in sign (multiples of π being neglected), by which the direction δ is to be rotated in order to become parallel to the direction D. If D_i ($i=1, \cdots, p$) are p given directions, the equation $\sum_{i=1}^{p}(\delta, D_i)=0$ determines p directions δ which make an angle π/p with each other. The author calls these directions the p-sectors of the given directions D_i. He considers the curves of class p such that the p-sectors of any set of concurrent tangents have fixed directions. It is shown that a necessary and sufficient condition for a curve to have this property is that the line at infinity shall be a double tangent of the curve, the points of contact being the circular points at infinity. The degree of a curve belonging to the family considered is comprised between $2p-2$ and p^2-p-2, its isotropic tangents all coincide with the line at infinity, etc. *N. A. Court* (Norman, Okla.).

Oberdorfer, G. Die kissoidale Erzeugung der zirkularen Kubik und bizirkularen Quartik aus der komplexen Ortskurvendarstellung. Österreich. Ing.-Arch. 2, 309–316 (1948).
In the study of alternating currents the problem presents itself of constructing a circular cubic, or a bicircular quartic, from its complex (vectorial) equation. The author shows that either curve can be constructed as a cissoid [Harold Hilton, Plane Algebraic Curves, Oxford, 1932] of a straight line and a circle for the cubic, and two circles for the quartic. Drawings illustrate the construction. *N. A. Court.*

Sz. Nagy, Gyula. Über die allgemeinen Lemniskaten. Acta Univ. Szeged. Sect. Sci. Math. 11, 207–224 (1948).
A general lemniscate $L_n(\rho)$ of radius ρ is defined to be the locus of a point in the Euclidean plane whose distances from n fixed points have the constant product ρ^n. The fixed points are called foci (Mittelpunkte); some of them may coincide, but the distinct foci are called kernels. The minimum and maximum distances between kernels are denoted by δ and Δ. The radius of the smallest circle enclosing the foci is r. A connected piece of the curve is called a branch (Zug). A simple closed curve occurring as a branch or part of a branch is called a cycle. Thus Bernoulli's lemniscate is the $L_2(\rho)$ for which $\rho=r=\frac{1}{2}\delta=\frac{1}{2}\Delta$; being a "figure of eight," this has two cycles forming one branch.
The author establishes many properties of the general lemniscate, such as the following. The smallest convex region enclosing the foci is penetrated by every normal and encloses all the singular points (if any occur). Each cycle encloses at least one kernel. If $\rho<\frac{1}{2}\delta$, there is a branch for each kernel. On the other hand, if $\rho \geqq \Delta$, the lemniscate consists entirely of one branch; and if $\rho \geqq r(1+\operatorname{cosec} \pi/8)$ it is a convex curve. *H. S. M. Coxeter* (New York, N. Y.).

Abramescu, N. Sur les quartiques gauches de première espèce. Acad. Roum. Bull. Sect. Sci. 27, 105–107 (1947).
Étude analytique du lieu des points S de l'espace dont les rapports mutuels des distances aux 3 côtés d'un triangle plan $A_1A_2A_3$ sont fixés. Ce lieu, défini par l'égalité de carrés de rapports d'aires, est une quartique gauche de première espèce, située sur 3 cônes du deuxième degré de sommets respectifs A_1, A_2, A_3, et sur un cylindre hyperbolique équilatère de génératrices perpendiculaires au plan du triangle. Les 4 points d'intersection de la quartique avec un plan parallèle au plan du triangle sont donc sur une hyperbole équilatère. Les 4 points d'intersection de la quartique avec le plan du triangle s'y construisent aisément.
P. Belgodère (Paris).

Kerawala, S. M. Poncelet porism in two circles. Bull. Calcutta Math. Soc. 39, 85–105 (1947).

Two circles of radii R, r and centres d apart are circumscribed and inscribed to the same n-gon. The author discusses Poncelet's problem of finding the "porism" between R, r, d in both rational and irrational forms. He considers the transformations T_r by which we pass from n to rn, conjecturing a very simple form for T_p when p is prime, and he includes two connecting rational and irrational forms. He builds up a number of basic expressions in terms of which he obtains porisms for n reaching into three figures. The author makes use of two papers by the reviewer [Proc. London Math. Soc. (2) 22, 104–123 (1923); 25, 17–44 (1925)], correcting some errors therein but going beyond to many forms that must certainly be new.

T. W. Chaundy (Oxford).

Deaux, R. Sur trois homographies du plan de Gauss. Bull. École Polytech. Jassy [Bul. Politehn. Gh. Asachi. Iași] 2, 106–116 (1947).

Outre le cas classique de 3 homographies ayant le même couple de points doubles, c'est-à-dire appartenant à un faisceau linéaire dont fait partie l'homographie identique, il existe une autre possibilité, déjà étudiée, approfondie ici par le calcul (portant sur les nombres complexes, affixes des points du plan de Gauss), d'homographies binaires, superposées, telles que le birapport formé par tout élément Z et ses 3 homologues Z_1, Z_2, Z_3 ait une valeur constante. Les 3 homographies ont chacune 2 points doubles distincts, dont chacun est double pour 2 d'entre elles, et ont un et un seul couple commun P, Q de points homologues distincts, ce qui équivaut au fait que le produit de leurs invariants est égal à l'unité. La constance du birapport des affixes Z, Z_1, Z_2, Z_3 entraîne, dans le plan de Gauss, des propriétés géométriques communes à tous les quadrangles représentatifs.

P. Belgodère (Paris).

Tigano, O. Sulle omografie il cui quadrato è una simmetria rispetto ad un punto. Matematiche, Catania 2, 108–123 (1947).

Les seules homographies de l'espace projectif S_n dont le carré soit une homologie harmonique (par rapport à un point et un hyperplan à $n-1$ dimensions) sont les homographies régulières (c'est-à-dire dont les équations sont réductibles à la forme $x_i' = \lambda_i x_i$) dont les invariants absolus (rapports des λ_i au dernier d'entre eux λ_{n+1}) aient tous pour carré (-1). Pour n impair (par exemple pour la droite ou l'espace à 3 dimensions), il n'existe pas, dans l'espace réel S_n, de telle homographie qui soit réelle. Pour $n=2$, on a les homographies des types: (1) $x'=\epsilon ix$, $y'=\epsilon iy$, $t'=t$, avec $\epsilon=\pm 1$, $i^2=-1$ (homothétie imaginaire); (2) $x'=hx-\lambda y$, $y'=(1+h^2)x/\lambda-hy$, $t'=t$.

P. Belgodère (Paris).

Mandzyuk, Anastasiya. On the construction of systems of lines in an n-dimensional space. Doklady Akad. Nauk SSSR (N.S.) 61, 609–611 (1948). (Russian)

A "bioplane" is an $(n-2)$-dimensional plane in a space of n dimensions. A system of bioplanes $S'_{n-1}, S''_{n-1}, \cdots, S^{(k)}_{n-1}$ in a space S_{n+1} is said to be reduced if each one lies in a corresponding hyperplane belonging to a pencil. The first, fairly obvious, theorem is that $S'_{n-1}, \cdots, S^{(k)}_{n-1}$ will form a reduced system in S_{n+1} if there exists an S_{n-1} which intersects the system of bioplanes in planes of dimension S_{n-2}. From this it follows that S_2', S_2'', S_2''' in S_4 always form a reduced system. Another theorem is that if from an arbitrary point O of S_{n-1} all lines intersecting $S'_{n-1}, \cdots, S^{(k)}_{n-1}$ (a reduced system) are projected on a hyperplane we shall obtain ∞^{2n-k}, $3 \leqq k \leqq 2n$, lines intersecting $k+1$ hypersurfaces of S_n in projective ranges of points. If the aggregate of lines in this theorem is denoted by Σ_n^k, then Σ_3^3 is the ordinary tetrahedral complex. A generalization of this is Σ_n^n which contains $n+1$ bundles of lines each containing ∞^{n-1} lines.

M. S. Knebelman (Pullman, Wash.).

Cartan, Élie. Sur l'espace anallagmatique réel à n dimensions. Ann. Soc. Polon. Math. 20 (1947), 266–278 (1948).

On peut représenter une transformation anallagmatique réelle T de l'espace à n dimensions par une substitution linéaire déterminée S (ou, plus exactement, par 2 substitutions linéaires) portant sur les coordonnées anallagmatiques, à $n+1$ dimensions, des hypersphères, et laissant invariante la forme quadratique dont l'annulation correspond aux hypersphères-points. Les hypersphères improprement réelles, les hypersphères-points, et les hypersphères proprement réelles orientées, sont susceptibles de coordonnées "semi-normales," positivement homogènes. Toute transformation anallagmatique réelle T est alors représentable par une et une seule substitution linéaire, bien déterminée, faisant passer d'un système de coordonnées semi-normales d'une hypersphère à un système de coordonnées semi-normales de l'hypersphère transformée.

L'invariant anallagmatique quantitatif formé par le "produit scalaire" de deux hypersphères "unitaires" ne suffit pas pour caractériser la position anallagmatique qualitative de deux hypersphères proprement réelles orientées, non sécantes, lorsque l'on interprète l'orientation analytique d'une hypersphère proprement réelle par la distinction entre domaine intérieur et domaine extérieur.

La détermination de $p \leqq n+1$ hypersphères unitaires proprement réelles, orientées, linéairement indépendantes, connaissant leurs produits scalaires mutuels, est lié à la décomposition en carrés de la forme quadratique fondamentale qui représente le carré scalaire de l'hypersphère générale, dépendant de p paramètres homogènes, du système linéaire engendré par ces hypersphères. Pour p carrés positifs, le problème est possible, et toutes ces solutions sont anallagmatiquement égales entre elles. Pour $p-1$ carrés positifs et 1 ou 0 carré négatif, il y a 2 familles de solutions, anallagmatiquement égales entre elles dans chaque famille, mais anallagmatiquement distinctes quand on passe d'une famille à l'autre.

P. Belgodère (Paris).

Convex Domains, Integral Geometry

Pogorelov, A. V. The uniqueness of closed tubes. Uspehi Matem. Nauk (N.S.) 3, no. 3(25), 170–182 (1948). (Russian)

In E^3 let C_i be a closed convex curve of class C''' and nonvanishing curvature in the plane P_i, $i=1, 2$, where P_1 (P_2) does not intersect C_2 (C_1). The author calls the boundary T of the convex closure of $C_1 \cup C_2$ a tube. If T' is a convex surface intrinsically isometric to T, then T' is congruent to T; that is, T' can be moved to coincide either with T or the image of T under a reflection of E^3 in a plane.

H. Busemann (Los Angeles, Calif.).

Tietze, Heinrich. Über die Anzahl der stabilen Ruhelagen eines Würfels. Elemente der Math. **3**, 97–100 (1948).

Given a plane convex domain whose boundary B can be represented in polar coordinates by a function $r=f(\theta)$ with only finitely many maxima and minima, suppose, furthermore, that there is a homogeneous mass distribution in the inside and that B does not contain any circular arc with the center in the center of gravity. In this case the figure has at least two stable positions of equilibrium. *W. Feller.*

Santaló, L. A. On plane hyperconvex figures. Summa Brasil. Math. **1** (1946), no. 11, 221–239 (1948). (Spanish. English summary)

Let E be a differentiable closed strictly convex curve with a center (in the Euclidean plane). Call E-curves all curves originating from E by translations. According to A. E. Meyer a set H is E-hyperconvex if any two points a, b of H can be connected by an arc of an E-curve and if every E-curve arc connecting a and b lies in H. Assume from now on that H is the closure of a simply connected bounded open set. Then it is E-hyperconvex if and only if no E-curve lies in H and the boundary of H is not intersected by any E-curve in more than two points, unless it contains an arc of an E-curve. It is then convex in the usual sense. If $A(H)$ and $A(E)$ denote the areas of H and E and $A(H, E)$ is the mixed area of H and E then an E-hyperconvex H satisfies the relation $A(H, E)\leqq\frac{1}{2}\{A(H)+A(E)\}$; the equality sign holds only if H is an E-curve.

Let C_r denote a circle of radius r. Then the isoperimetric deficit of a C_r-hyperconvex H with length $L(H)$ satisfies the relation $L^2(H)-4\pi A(H)\leqq\{2\pi r-L(H)\}^2$. If H is C_r-hyperconvex and $O(\xi, \eta)$ is the center of a circle with radius r which intersects H, then the intersection is a circular arc. Let ω be the angle subtended by this arc at $O(\xi, \eta)$. Then $\int\omega d\xi d\eta=2\pi A(H)$ and $\int\sin\omega d\xi d\eta=2\pi A(H)-A^2(H)r^{-2}$. Several extremum problems are solved, for instance: among all C_r-hyperconvex curves whose circumscribed circle has a given radius R, the equilateral triangle consisting of three circular arcs with radius r inscribed to C_R has minimal diameter. *H. Busemann* (Los Angeles, Calif.).

Gal, I. S. A theorem concerning regular polygons. Bull. École Polytech. Jassy [Bul. Politehn. Gh. Asachi. Iaşi] **3**, 98–106 (1948).

The author proves the following results (generalizing and sharpening several previous ones by various authors). Let $A_1, \cdots, A_n$ be the vertices of a regular polygon, P any point. Then

$$\sum_{k=1}^{n}\overline{A_kP}\geqq\{(\cos\tfrac{1}{2}\pi/n)^{\frac{1}{2}[1+(-1)^n]}\csc\tfrac{1}{2}\pi/n\}\max_i\overline{A_iP}+\tfrac{1}{2}[1+(-1)^n]\min_i(\overline{A_iP}).$$

This result is best possible. *P. Erdös* (Syracuse, N. Y.).

Hadwiger, H., Glur, P., und Bieri, H. Die symmetrische Kugelzone als extremaler Rotationskörper. Experientia **4**, 304–305 (1948).

Let V, F and M be the volume, the surface area and the total mean curvature of a convex body in ordinary space. These invariants are restricted by the well-known Minkowski inequalities $4\pi F\leqq M^2$ and $3VM\leqq F^2$. It is furthermore known that they must satisfy another inequality giving the minimum of M for prescribed V and F. This unknown inequality seems to be complicated. The authors announce the following result. If only bodies of revolution are considered, the minimum of M is attained by the symmetrical spherical zone. The corresponding inequality is transcendental. *W. Fenchel* (Copenhagen).

Hadwiger, H. Sur le déficit isopérimétrique d'un polygone formé par des arcs de cercle. Gaz. Mat., Lisboa **9**, no. 36, 1–2 (1948).

Let P be a convex polygon in the Euclidean plane with vertices p_i, $i=1, \cdots, n+1$; $p_{n+1}=p_1$. On each side p_ip_{i+1} describe a circular arc C_i outside of P and such that for $q_i\in C_i$ the sum of the angles $p_iq_ip_{i+1}$ and $p_{i+1}q_{i+1}p_{i+2}$ is always less than 2π. Then $\cup C_i$ is a Jordan curve C. If L denotes the length of C and F the area of the domain bounded by C, then for given P the isoperimetric deficit $L^2-4\pi F$ is minimal if and only if the arcs C_i are chosen as follows: let P_0 be a convex polygon $p_1'\cdots p_{n+1}'$ with $p_i'p_{i+1}'=p_ip_{i+1}$ and inscribed in a circle K; the arc C_i is congruent to the arc of K over $p_i'p_{i+1}'$. *H. Busemann.*

Hadwiger, H. An integral mean value for the Euler characteristic for movable ovals. Revista Unión Mat. Argentina **13**, 66–72 (1948). (Spanish)

Let "oval" denote a bounded closed convex set in the plane. The Euler characteristic of the union U of a finite number of ovals is the number of outer minus the number of inner contours of U. Let H be an oval and consider n variable ovals K_i, $i=1, \cdots, n$, congruent to a fixed oval K. The mean value of the Euler characteristic of the intersection of $\cup K_i$ with H over all positions of the K_i for which each K_i intersects H is

$$1+(n-1)[1-n\pi F_HL^2(LL_H+2\pi F_H)^{-2}]\times(LL_H+2\pi F_H)(LL_H+2\pi F+2\pi F_H)^{-1},$$

where L, L_H are the lengths and F, F_H the areas of K and H, respectively. *H. Busemann* (Los Angeles, Calif.).

Fejes Toth, L., und Hadwiger, H. Ueber Mittelwerte in einem Bereichsystem. Bull. École Polytech. Jassy [Bul. Politehn. Gh. Asachi. Iaşi] **3**, 29–35 (1948).

Proof of a theorem stated previously [Experientia **3**, 366–369 (1947); these Rev. **9**, 247]. *H. Busemann.*

Algebraic Geometry

Bureau, Florent. Remarque sur des sommes d'intégrales abéliennes attachées à certaines courbes algébriques. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **34**, 287–289 (1948).

Announcement of results concerning addition theorems which can be derived from Huygens' principle for the elementary solution of a partial differential equation with constant coefficients. *F. John* (New York, N. Y.).

Burniat, Pol. Surfaces canoniques quadruples. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **34**, 239–251 (1948).

Continuing his investigations into the construction of quadruple canonical surfaces for all values of p_g, the author is led to the study of complete linear systems of curves in a plane which are compounded of an involution, and obtains a number of theorems, too long to quote here, which ensure the accuracy of his constructions. *D. Pedoe* (London).

Nollet, Louis. La classification et la construction des courbes non hyperelliptiques de genre six. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **34**, 389–398 (1948).

Enriques and Chisini have remarked that an algebraic curve of genus 6 possesses, in general, a g_4^1 without fixed points [Lezioni sulla Teoria Geometrica delle Equazioni e delle Funzioni Algebriche, v. 3, Zanichelli, Bologna, 1924, p. 103]. In order to demonstrate the possible meaning which can be attached to this assertion, the author classifies all algebraic curves of genus 6 which are not hyperelliptic. He obtains five types which can be characterised by the linear series of sets of points which they contain as follows: (1) contains a g_5^2, a series of g_4^1 of genus 6; (2) contains a g_3^1 and a g_4^1; (3) contains a g_3^1 but no g_4^1; (4) contains no g_3^1 or g_3^1 and contains from one to five g_4^1; (5) contains an elliptic series of g_4^1, an elliptic involution of pairs of points.

D. Pedoe (London).

Nollet, Louis. Quelques surfaces algébriques qui sont, de plusieurs manières, enveloppes de systèmes de surfaces. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **34**, 236–238 (1948).

The surfaces in question have equation

$$X_1^3+X_2^3+X_3^3-2X_2X_3-2X_3X_1-2X_1X_2=0,$$

where X_i is a product of general homogeneous forms in (x_0, x_1, x_2, x_3) of total degree m.

J. A. Todd.

Nollet, Louis. Les surfaces algébriques pour lesquelles $p^{(1)}<3p_a$ et le nombre-base des surfaces irrégulières. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **34**, 426–431 (1948).

An algebraic surface for which $p^{(1)}<3p_a$ is either (i) regular and free from torsion or (ii) contains a pencil, of genus equal to the irregularity of the surface, of irreducible hyperelliptic curves.

J. A. Todd (Cambridge, England).

Dantoni, Giovanni. Sulle superficie algebriche con infinite involuzioni irregolari. Pont. Acad. Sci. Comment. **7**, 383–396 (1943).

L'auteur détermine les surfaces algébriques F, d'irrégularité q, possédant des infinités d'involutions I_n de groupes de n points dont les images F' sont d'irrégularité $q'\leqq q$. Cette étude apporte un complément aux travaux de Severi [Ann. Mat. Pura Appl. (4) **21**, 1–20 (1942); ces Rev. **6**, 17] limités à $q'=q$. Les surfaces F peuvent se classer selon la nature de F'. (a) Si F' possède un faisceau de genre q', F possède un faisceau de genre q', composé avec I_n, qui peut varier dans un système continu infini. (b) Si F' est d'irrégularité 2 représentable sur une V_2 de Picard, F ne peut posséder un système continu d'involutions. (c) Si F' est une surface elliptique, possédant un faisceau elliptique et un autre de genre $q'-1$, F possède des faisceaux de mêmes genres, composés avec I_n, qui ne peut varier dans un système continu. (d) Si F' ne rentre pas dans les cas précédents, l'auteur montre que la série d'irrégularité de Severi contient une série $\infty^{q'-1}$ individualisée et composée par I_n ou son intermédiaire; les I_n sont alors en nombre fini et leur ordre $n\leqq I+4$, I étant l'invariant de Zeuthen-Segre de F.

En conséquence, toute surface possédant un système infini Σ continu irréductible d'involutions d'irrégularité $q'>0$, a un faisceau de genre q' composé avec toutes les involutions de Σ. Ces involutions de genre q' sont birationnellement identiques à des réglées de genre q' ou à des surfaces d'irrégularité q' avec un faisceau de genre q' de courbes elliptiques. Si F n'a pas de faisceau mais des systèmes infinis discontinus de I_n, F équivaut à une surface elliptique simple ou multiple. Les I_n sauf un nombre fini sont alors équivalentes birationnellement à des surfaces elliptiques. Enfin une surface privée de faisceau irrationnel ne peut avoir qu'un nombre fini d'involutions d'irrégularité supérieure à 2.

B. d'Orgeval (Grenoble).

Dantoni, Giovanni. Superficie algebriche con infinite involuzioni irrazionali e involuzioni ∞^{2r} con $r>1$. Pont. Acad. Sci. Comment. **9**, 169–203 (1945).

L'auteur reprend l'étude des surfaces algébriques F possédant des involutions I_n, l'étendant au cas où la F' image de I_n est irrationnelle et régulière. Considérant les formes tensorielles de première espéce liées à F' il montre qu'au système k-canonique de F' correspond sur F un système linéaire composé avec I_n et invariant si I_n varie continument; d'où $(p^1)'\leqq 1$ et $P_k'\leqq 1$. Si donc F posséde un systéme continu de I_n, les F' qui les représentent ont leurs plurigenres $\leqq 1$; elles appartiennent à des types déterminés par Enriques [Atti Accad. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (5) **23**, 206–214, 291–297 (1914)] et Geppert [Jber. Deutsch. Math. Verein. **31**, 18–39 (1931)]. Sans tenir compte de la correction de Geppert, qui semble d'ailleurs être sans consequence sur le résultat, l'auteur indique les types possibles de F': (a) réglées de genre p, (b) elliptiques de genre $p_g=0$, (c) surface d'Enriques $P_2=1$, (d) surfaces régulières de genres 1. Apparait impossible le cas des F' hyperelliptiques de rang 1. A (a) correspondent les F dotées d'un faisceau de genre p; à (b) les surfaces F représentables simplement ou multiplement sur la surface des couples de points de deux courbes elliptiques; à (c) et (d) correspondent des surfaces de Picard.

Si l'on suppose que le système infini des I_n est discontinu, la condition $p^1\leqq 1$ se conserve, mais non celle des plurigenres. L'auteur montre que si F contient une I_n irrationnelle avec $p^1>1$ sur F et F' il existe un système k-canonique dont la partie variable est irréductible et simple, d'où il déduit que F ne peut contenir qu'un nombre fini de telles I_n. Les I_n possibles sont donc équivalentes (a) aux réglées, (b) aux elliptiques d'irrégularité quelconque, (c) aux surfaces de Picard, (d) aux surfaces d'irrégularité q dotées d'un faisceau, de genre q, de courbes elliptiques, (e) aux surfaces réguliéres de genres 1. Tous ces cas sont possibles. On peut en déduire les résultats donnés par l'auteur dans la note précédente.

Passant à l'étude des involutions ∞^{2r} $(r>1)$ de groupes de n points sur une surface algébrique F, J_n^{2r}, l'auteur associe, à cette J, ∞^{2r-2} involutions I_m $(m=n-r+1)$ obtenues en fixant $r-1$ points de F et considérant tous les groupes de J les contenant; ces I_m associées sont toutes rationnelles. Tout cycle linéaire lieu de groupes de la J est nul ou pseudo-nul, tout cycle bidimensionnel de tels groupes est algébrique ou pseudo-nul. Une J_n^{2r} est donc série de pseudo-équivalence; une I_m associée, série d'équivalence.

B. d'Orgeval (Grenoble).

Kadeřávek, František. Sur les surfaces du 4° formées par la sommation de deux surfaces du 2°. Věstník Královské České Společnosti Nauk. Třída Matemat.-Přírodověd. **1947**, no. 4, 10 pp. (1948). (Czech. French summary)

I. La surface ϵ est la somme d'une surface ϵ' du 2° et d'une surface cylindrique ϵ'' touchant ϵ' aux deux sommets conjugés. Par chaque point de la surface ϵ passent 4 coniques.

II. La surface ϵ est la somme de deux hyperboloides ayant le même centre et les mêmes axes. Les surfaces coniques asymptotiques se touchent le long de deux droites et les hyperboloides ϵ', ϵ' ont deux sommets communs. Par chaque point de la surface ϵ on peut mener sept courbes du 2°. III. Par la sommation de deux cônes du 2° homothétiques ayant une axe commune on forme une surface spéciale du 4° dont six coniques passent par chaque point.

Author's summary.

Bishara, S., and Amin, A. Y. The configuration of Schur quadrics and the parabolic curve of the trinodal cubic surface. Amer. J. Math. **70**, 414–422 (1948).

This paper deals with the limiting forms, for a trinodal cubic surface F^*, of certain known results for a general cubic surface F. The 36 Schur quadrics of F are shown to reduce to 14 for F^*, and the configuration of these is exhibited. If 9 lines of F are the intersection of a pair of Steiner trihedrals, they are the base of a pencil of surfaces F whose parabolic curves generate a quartic surface Γ [Bishara, Proc. London Math. Soc. (2) **35**, 241–248 (1933)]. For an analogous pencil of surfaces F^* the locus Γ breaks up into a quadric cone P (with vertex at the fourth node of the unique 4-nodal surface of the pencil) and the plane of the nodes (counted twice). The relations of P to the F^* are discussed. *J. G. Semple* (London).

Zappa, Guido. Sull'esistenza di curve algebricamente non isolate, a serie caratteristica non completa, sopra una rigata algebrica. Pont. Acad. Sci. Acta **7**, 1–4 (1943).

Severi has shown [Ann. Mat. Pura Appl. (4) **23**, 149–181 (1944); these Rev. **7**, 477] that the characteristic series of a generic irreducible curve of an algebraic system which is at least ∞^1 in an algebraic surface is always complete. A necessary condition for a particular curve $\bar{C}$ of an algebraic system $\{C\}$ which is ∞^r ($r \geqq 1$) to have an incomplete characteristic series is that the number of distinct curves of $\{C\}$ which pass through r generic points of $\bar{C}$ be less than the number of curves of $\{C\}$ which pass through r generic points of the surface. When this is the case, at least two of the curves of $\{C\}$ which pass through r generic points of the surface tend to $\bar{C}$ as the r generic points tend to r generic points of $\bar{C}$. The author constructs an algebraic curve on a scroll of genus 2 and order 6 in S_3 which belongs to an algebraic ∞^1 system, but has an incomplete characteristic series. The necessary conditions given by Severi are verified. *D. Pedoe* (London).

Zappa, Guido. Su alcuni contributi alla conoscenza della struttura topologica delle superficie algebriche, dati dal metodo dello spezzamento in sistemi di piani. Pont. Acad. Sci. Acta **7**, 4–8 (1943).

In this note the author reports further results obtained for algebraic surfaces by degenerating them into sets of planes [cf. Atti Accad. Italia. Mem. Cl. Sci. Fis. Mat. Nat. **13**, 989–1021 (1942); these Rev. **8**, 527]. A full account is to be published elsewhere. One feature is an attempt to find a topological significance for the geometrical genus of an algebraic surface. This problem has already been solved by Hodge [J. London Math. Soc. **8**, 312–319 (1933)] as the author acknowledges, but he claims to have found another topological interpretation which is mathematically significant. *D. Pedoe* (London).

Zappa, Guido. Sull'esistenza, sopra le superficie algebriche, di sistemi continui completi infiniti, la cui curva generica è a serie caratteristica incompleta. Pont. Acad. Sci. Acta **9**, 91–93 (1945).

A construction is given, on a scroll of genus 2 and order f in S_3, of a complete algebraic ∞^1 system of curves which is such that the generic curve has an incomplete characteristic series. *D. Pedoe* (London).

Roth, L. Sulle forme che contengono una data varietà algebrica. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **3**, 541–545 (1947).

L'auteur indique deux critères permettant de décider si une variété algébrique irréductible V_d^n de S_r ($d<r-1$) peut appartenir à une forme V_{r-1}^n, d'ordre n donné inférieur à $m-r+d+1$. Ces critéres sont: (1) V_d est normale dans S_r et ses sections hyperplanes sont sur des variétés V_{r-2}^n, telles que $n-1$ d'entre elles puissent être sections hyperplanes d'une V_{r-1}^n; si r est impair, la réponse est affirmative, quel que soit n; si r est pair, elle est affirmative si $n \leqq r+1$, sinon on ne peut rien dire; (2) les formes d'ordre s coupent sur V_d un système linéaire complet et les sections hyperplanes de V_d sont sur des V_{r-2}^n, telles que $n-s$ d'entre elles ($n \leqq r+s$) forment des sections hyperplanes d'une V_{r-1}^n; dans ce cas la réponse est affirmative.

L'auteur part du cas simple des V_d normales dont les sections hyperplanes appartiennent à des quadriques V_{r-2}^2; la propriété s'établit par calcul de la dimension du système découpé sur V_d par les V_{r-1}^2 passant par une V_{r-2}^2. Ceci le conduit à construire dans le cas général un système de V_{r-1}^n, passant par des V_{r-2}^n, coupant le systéme des sections hyperplanes et de dimension supérieure à r. Ceci conduit aux critères indiqués. *B. d'Orgeval* (Grenoble).

Bílek, Jan. Certain properties of sextics with double points, derived with the aid of Cremona transformations. Věstník Královské České Společnosti Nauk. Třída Matemat.-Přírodověd. **1947**, no. 5, 10 pp. (1948). (Czech)

Bílek, Jan. Sur une involution du plan J_{11} de la deuxième classe. Časopis Pěst. Mat. Fys. **73**, 17–30 (1948). (Czech. French summary)

This is the well-known involution generated by the pair of free intersections of a conic and a cubic from fixed pencils, of which the 4 base points of the one are included among the 9 base points of the other [cf. H. P. Hudson, Cremona Transformations in Plane and Space, Cambridge University Press, 1927, p. 126]. It may also be generated, as the author shows here, by an ∞^3 system of C_8 which have 4 double base points and 5 simple base points, the 9 being an associated set; for such a system is compounded of a J_{11}, and the coincidence curve (of order 7) of J_{11} is the locus of extra nodes of irreducible curves of the system. By making various sets of three fundamental points collinear, the author obtains involutions of every order less than 11.

J. G. Semple (London).

Villa, Mario. Un fascio di quintiche collegato alle trasformazioni puntuali. Boll. Un. Mat. Ital. (3) **3**, 8–15 (1948).

Considérant une transformation ponctuelle entre deux plans, l'auteur associe à une direction d'inflexion issue de O, une cubique E_3, d'osculation en O; sur la tangente à celle-ci, il existe un point pôle de toute droite issue de O par rapport aux coniques osculatrices en O à E_3; le lieu de ce point, si la droite varie, est une quintique ayant en O un point

quadruple dont les tangentes sont la tangente à E_3 et les trois directions caractéristiques de la transformation. Ces quintiques forment un faisceau; celles correspondant aux directions caractéristiques se décomposent en une droite et une quartique ayant un point triple en O dont les tangentes sont les directions caractéristiques.

L'existence de ces quartiques permet d'associer à la transformation un système de référence intrinséque. Dans le plan de O ce seront: (1) trois droites caractéristiques; celles-ci et une bitangente à Q_4; (2) deux droites caractéristiques; celles-ci, une droite joignant O à une inflexion et la tangente d'inflexion; (3) une droite caractéristique; celle-ci, les droites joignant O aux inflexions et une de ces tangentes inflexionnelles. Dans le plan correspondant, le système sera donné par les transformés dans les projectivités caractéristiques. *B. d'Orgeval* (Grenoble).

Villa, Mario. Ricerche locali sulle trasformazioni cremoniane. II. Mem. Accad. Sci. Ist. Bologna. Cl. Sci. Fis. (10) **2**, 117–126 (1946).

For part I cf. same Mem. Cl. Sci. Fis. (10) **1**, 189–200 (1944); these Rev. **9**, 464.

Differential Geometry

Čebyšev, P. L. On the cutting of garments. Uspehi Matem. Nauk (N.S.) **1**, no. 2(12), 38–42 (1946). (Russian)

A summary appeared in Assoc. Française Avancement Sci. C. R. **7**, 154–155 (1878); the full text of the lecture was not published in Čebyšev's collected works. This translation of the French manuscript was published in Arhiv Istorii Nauki i Tehniki Akad. Nauk SSSR **9**, 347–352 (1936).

Inzinger, Rudolf. Über eine Abbildung der Speere einer Ebene. Monatsh. Math. **52**, 124–137 (1948).

Let (x, y) be rectangular Cartesian coordinates in a plane π. Let h and φ denote the polar coordinates of a directed straight line l in π; thus h (positive, negative or zero) is the length of its normal from the origin o, and φ is the angle between the x-axis and that normal. Capital letters denote the corresponding quantities in a second plane Π. Through $\mathfrak{A}$: $H=h^2$, $\Phi=2\varphi \pmod{2\pi}$, each L in Π is mapped on a pair t of nonoriented straight lines in π symmetric with respect to o. By introducing homogeneous line coordinates, $\mathfrak{A}$ can be extended to isotropic L's. A curve in Π (considered as the set of its oriented tangents) is mapped through $\mathfrak{A}$ on a nonoriented curve in π symmetric with respect to o. In particular, $\mathfrak{A}$ maps the oriented circles C of Π on the conics c with center o. Let R (positive, negative or zero) be the radius of C and (X, Y) its center. The "tangential distance" of C_1 and C_2 is the length

$$\delta=[(X_1-X_2)^2+(Y_1-Y_2)^2-(R_1-R_2)^2]^{\frac{1}{2}}$$

of their common directed tangents. In π, δ can be interpreted as the "tangential moment" of the corresponding pair of c's. In this fashion, π becomes a model of Laguerre geometry.

The Laguerre projection maps the C's in Π on the points (X, Y, R) of a pseudo-Euclidean 3-space R_3 with metric δ. Combining this projection with $\mathfrak{A}$, we obtain a mapping $\mathfrak{B}$ of the manifold of the c's on R_3. Through $\mathfrak{A}$, the various transformation groups of the L's in Π are isomorphic to groups of the t's in π. Similarly, $\mathfrak{B}$ determines isomorphisms between groups in R_3 and transformation groups of the manifold of the c's. *P. Scherk* (Saskatoon, Sask.).

Buzano, Piero. Corrispondenze fra curve piane ed evolventi proiettive. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **81–82**, 102–108 (1948).

On the tangents to a plane curve $A(t)$, take three points $A_i(t)$; the tangents to their trajectories determine a triangle of vertices $P_i(t)$; the tangents to the trajectories of the P_i meet in a point $R(t)$. This configuration, which arose in a previous paper of the author [same Atti Cl. Sci. Fis. Mat. Nat. **75**, 288–295 (1940); these Rev. **3**, 87], is here studied with the methods of É. Cartan [Leçons sur la Théorie des Espaces à Connexion Projective, Gauthier-Villars, Paris, 1937, particularly chap. 2]. Some use is made of allied results of A. R. Jerbert [Amer. J. Math. **51**, 99–108 (1929)]. By using projective normalizations and the P_i as local reference system, invariants are found and interpreted. In this connection Cartan's projective evolutes play significant roles. *J. L. Vanderslice* (College Park, Md.).

Buzano, Piero. Osservazioni intorno agli invarianti proiettivi di elementi curvilinei. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **81–82**, 109–113 (1948).

In order that a curve C^n of S_n be rational normal, is it both necessary and sufficient that all projective invariants of three E_{n-1}'s (curvilinear elements of order $n-1$) belonging to C^n be constant? The necessity part was proved by M. Ales [Rend. Circ. Mat. Palermo **63**, 111–112 (1942); these Rev. **9**, 199]. The sufficiency part when $n=2$ was proved by the author [Boll. Un. Mat. Ital. (2) **3**, 201–207 (1941); these Rev. **3**, 182; **7**, 620]. The present paper constitutes a preliminary attack on the sufficiency part for $n=3$. The author determines the six invariants of three E_2's in S_3 and finds their projective and metric significance. *J. L. Vanderslice* (College Park, Md.).

Buzano, Piero. Invarianti proiettivi di due elementi differenziali curvilinei. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **81–82**, 114–121 (1948).

The author considers two curvilinear differential elements of the second order in S_3. Each element defines a center, tangent line and osculating plane. Assuming these loci distinct there are still five cases to investigate depending on incidence relations. In each case the projective invariants of the configuration are determined and the most noteworthy given projective or metric interpretation. *J. L. Vanderslice* (College Park, Md.).

Su, Buchin. On the point of inflexion of a curve in a projective space. Ann. Mat. Pura Appl. (4) **26**, 177–197 (1947).

In the manner in which E. Bompiani has developed the projective differential geometry of a plane curve at a point of inflexion by the use of certain osculants associated with the inflexion [Boll. Un. Mat. Ital. (1) **5**, 118–120 (1926)], the author has developed a similar theory for a curve in a projective n-dimensional space S_n. Extensive use is made of the representable singularity of a plane curve, a notion introduced by the author and used in his earlier papers [Tôhoku Math. J. **45**, 239–244 (1938); J. Chinese Math. Soc. **2**, 139–151 (1940); these Rev. **2**, 299]. A singular point of a plane curve C is said to be of order m ($m\geqq 3$) when the tangent t_0 of C at the point has a contact of order $m-1$

with C. The singularity O is said to be representable if an algebraic curve C_m of order m can be determined having (1) a unique $(m-1)$-ple point $O_{m-1}^{(m)}$ with coincident tangents t and (2) a contact of order $2m$ with C at O. The point of intersection $O_{m+1}^{(m)}$ of the tangents t_0 and t as well as the line $l_{m-1}^{(m)}$ joining the points O and $O_{2m}^{(m)}$ are covariantly associated with the representable singularity. The use of the conditions that a singularity be representable serves the following purposes: (1) to characterize geometrically a reference pyramid at a point of inflexion of a curve in S_n, (2) to determine associated canonical equations of the curve in terms of $\frac{1}{2}(n^2-n-6)$ projective invariants. *P. O. Bell.*

Bol, Gerrit. Einige neue Ergebnisse aus der Differentialgeometrie der Raumkurven im dreidimensionalen projektiven Raum. Arch. Math. 1, 3–8 (1948).

The equations of a curve C in projective three-space $x^i=x^i(t)$ may be normalized so that $(x, x', x'', x''')=K\neq 0$. Then these functions satisfy a differential equation of the form $x''''=10ax''+2dx'+ex$. This is replaced by the equivalent linear system

$$x_1'=x_2+3ax,\quad x_2'=x_3+4ax_1+bx,\quad x_3'=3ax_2+bx_1+cx,$$

where $x_1=x'$, $b=d-5a'$, $c=e-3a''-b'+9a^2$. If the coordinates $X^i(t)$ of the osculating plane to C are similarly normalized, then these coordinates satisfy a system of equations obtained from the above by changing b to $-b$. The functions b and c are seminvariants. If $b=0$, the tangents to C belong to a linear complex, the curve C being then called a "complex curve." If $c=0$, C is called a "coincidence curve," and if $b=c=0$, the curve is a cubic. Let P_1, P_2, P_3 be three points of C, and through P_1, P_2, P_3 draw a cubic C_3 having at P_1, P_2 the same osculating plane as C. The limit of C_3 as P_2 and P_3 approach P_1 along C is called the "harmonic-C_3" of C at P_1. The vertex x_3 of the tetrahedron (x, x_1, x_2, x_3) describes the harmonic-C_3 of C for all allowable transformations of the parameter t ($t=f(t^*)$, $dt^*/dt=\varphi(t)$, $A=\frac{1}{2}\varphi'/\varphi$). The planes X_3 are the osculating planes to the harmonic-C_3. The positions of the vertices of the tetrahedron (x, x_1, x_2, x_3) depend on A. The locus of x_3 for all A and for all positions of the point x on C is a surface called the first harmonic-surface of C. Again the edge x_2, x_3 generates a congruence, one of whose focal surfaces is the first harmonic surface of C; the other surface is called the second harmonic-surface. If these two harmonic-surfaces coincide the curve C is a coincidence curve. It should be noted that this definition of a coincidence curve is not equivalent to the definition of a coincidence curve as a curve for which the Halphen points of C coincide with the points of C.

V. G. Grove (East Lansing, Mich.).

Bompiani, Enrico. Un vettore proiettivo-conforme nella geometria differenziale delle superficie. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. 81–82, 215–218 (1948).

Let a_{ijk}, a_{ij} be the fundamental apolar tensors of a surface in a projective three-space and $\mathfrak{D}_{ij}$ the Ricci skew symmetric tensor. Then the vector ψ^s (introduced by Čech) which occurs in (1) $\epsilon\mathfrak{D}_{rs}\psi^s a_{ik}^{\ \ j}\mathfrak{D}_{jl}=\nabla_r a_{ikl}-a_{ikl}J_r$, where

$$\epsilon=\operatorname{sgn}(a_{12}^2-a_{11}a_{22}),\quad J_r=(\partial/\partial u^r)\log[\tfrac{1}{2}a_{ijk}a^{ijk}]^{\frac{1}{2}},$$

does not change by the transformation ${}'a_{ijk}=\rho(u)a_{ijk}$, for [reviewer's remark] $\nabla_r{}'a_{ijk}-{}'a_{ijk}{}'J_r=\rho(u)(\nabla_r a_{ijk}-a_{ijk}J_r)$. The author proves this statement in two different ways and gives also a very simple proof of the Čech formula $K=-\frac{1}{3}\nabla_s\psi^s$ (which represents the integrability conditions of (1)) in a special parameter system (K is the Gauss curvature of a_{ij}). *V. Hlavatý* (Bloomington, Ind.).

Müller, Hans Robert. Über Striktionslinien von Kurven- und Geradenscharen im elliptischen Raum. Monatsh. Math. 52, 138–161 (1948).

Methods are given for investigating a surface (S) in a three-dimensional elliptic space and applied to the study of the lines of striction of a one-parameter family F of curves on (S). From an internal point of view, (S) is a Riemannian space; tensor calculus yields the internal properties of the lines of striction already known when the imbedding space is the Euclidean space (E_3). To derive the external properties of (S), the author uses Cartan's method of the moving repère. The points of the elliptic space are identified with normed quaternions. The repère is an (absolutely) autopolar oriented tetrahedron $R=(P_0, P_1, P_2, P_3)$; with it are associated two trihedrals T_1 and T_2 in (E_3) formed by the left and right direction vectors of the edges of R. The equations expressing the differential displacements of R, T_1 and T_2, the relations between the differential Pfaff forms appearing in them and the integrability conditions are written out. The choice of R (determination of the canonical repère) with respect to a surface (S): $P_0=\mathbf{P}_0(u_1, u_2)$ and the family F is fixed as follows: P_1 and P_2 lie in the tangent plane to (S) in P_0; if $dP_0=\alpha_1P_1+\alpha_2P_2$, $\alpha_2=0$ represents the differential equation of the curves of F. In the case of a ruled surface (S): $g=\mathbf{g}(u)$, to each generator is attached one repère R; the classical choice of R when the space is Euclidean [É. Cartan, La théorie des groupes finis et continus et la géométrie differentielle traitées par la méthode du repère mobile, Gauthier-Villars, Paris, 1937, pp. 48–53] is transposed to elliptic space; for the canonical repère P_0P_1 and P_2P_3 are the common perpendiculars of the two generators $g(u)$ and $g(u+du)$. *C. Y. Pauc* (Cape Town).

Backes, F. Sur les familles de surfaces dont les lignes de courbure sont découpées par les développables d'une congruence de droites. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 34, 252–259 (1948).

The results of the paper are embodied in the following theorem. Let P be a generic point of a surface. The lines l parallel to the lines joining P to a fixed point F generate a congruence on each line l of which there exists an infinity of points Q generating a family of surfaces such that the developables of the congruence cut each surface of the family in its lines of curvature. The method used is that of moving frames of reference. *V. G. Grove.*

Marcus, F. Sopra qualche classe di superficie R. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 4, 303–305 (1948).

The problems proposed are to determine all surfaces on which there exist curves whose osculating planes at each of their points pass through the projective normals at those points, and dually to determine all surfaces on which there exist curves along which the tangent planes to the surface form a developable whose edge of regression "incontra" the reciprocal of the projective normal. The differential equations of the surface are written in the Fubini form in both point and plane coordinates: $x_{uu}=\theta_u x_u+\beta x_v+px$, $x_{vv}=\gamma x_u+\theta_v x_v+qx$, and $\xi_{uu}=\theta_u\xi_u-\beta\xi_v+\pi\xi$, $\xi_{vv}=-\gamma\xi_u+\theta_v\xi_v+\chi\xi$. Then necessary and sufficient conditions that the given surfaces possess the desired properties are $\pi_v=\chi_u$ and $p_v=q_u$, respectively. The integrability conditions of these systems consequently imply that $\beta_v=\gamma_u$, the surfaces being therefore surfaces R.

It follows also that the projective normal, and its reciprocal, are W-congruences. *V. G. Grove* (East Lansing, Mich.).

Creangă, Ion. Sur une correspondance entre deux surfaces réglées. Bull. École Polytech. Jassy [Bul. Politehn. Gh. Asachi. Iaşi] 2, 341–346 (1947).

Soit Γ la courbe d'intersection de deux surfaces réglées R_1, R_2, et g_1, g_2 les génératrices de R_1, R_2 issues du point courant M_0 de Γ. La correspondance étudiée est celle obtenue en associant, sur g_1 et g_2, les points M_1 et M_2 tels que les plans tangents en ces points à R_1 et R_2 se coupent dans le plan osculateur en M_0 à Γ. L'auteur définit analytiquement la correspondance précédente. Il montre que sur deux génératrices associées quelconques (g_1, g_2) la transformation faisant passer d'un point M_1 à son homologue M_2 est une homologie (dégénérée si Γ est asymptotique pour l'une au moins des deux surfaces R_1, R_2), établit que le centre Ω de cette homologie est situé sur la caractéristique du plan (g_1, g_2), détermine le lieu de Ω, donne les conditions pour que Ω soit constamment le point caractéristique du plan (g_1, g_2), et examine les particularités qui se présentent lorsque R_1, R_2 sont tangentes le long de Γ et lorsque l'homologie (M_1, M_2) est dégénérée. *P. Vincensini* (Besançon).

Creangă, Ion. Sur la correspondance "T" entre deux surfaces réglées. Bull. École Polytech. Jassy [Bul. Politehn. Gh. Asachi. Iaşi] 3, 131–140 (1948).

Dans une note antérieur [voir l'analyse ci-dessus], l'auteur a étudié, au moyen d'un repère affine, une correspondance ponctuelle entre deux surfaces réglées R_1, R_2, dite correspondance T, associant, sur deux génératrices quelconques g_1, g_2 de R_1, R_2 se coupant en M_0 sur l'intersection Γ_0 de R_1 et R_2, les points P_1 et P_2 en lesquels les plans tangents à R_1 et R_2 coupent le plan osculateur π en M_0 à Γ_0 suivant une même droite δ. La correspondance T induit sur g_1 et g_2 une homologie, en général non dégénérée, ayant pour centre un certain point K de la caractéristique du plan (g_1, g_2). Il reprend et complète ici l'étude précédente, en substituant au repère affine un repère projectif local $[M_0, M_1, M_2, M_3]$, (M_1, M_2) étant un couple de points correspondants intrinsèquement déterminé sur g_1 et g_2, et M_3 un point de la tangente en M_0 à Γ_0 de détermination également intrinsèque. Il étudie la correspondance entre les tangentes à R_1, R_2 en deux points P_1, P_2 homologues dans T. Les couples de rayons homologues dans les faisceaux de tangentes de sommets P_1 et P_2 coupent δ suivant les couples de points conjugués d'une involution. Cette involution est parabolique pour tout couple (P_1, P_2) de (g_1, g_2), et se réduit à l'identité pour le couple particulier (M_1, M_2). La correspondance T fait correspondre à un réseau R de R_1 un réseau $\bar{R}$ de R_2. Les tangentes aux courbes correspondantes de R et $\bar{R}$ forment deux demi-quadriques coupant π suivant deux coniques bitangentes, l'un des points de contact étant M_0 et la tangente correspondante M_0M_3 (tangente en M_0 à Γ_0). Les demi-quadriques osculatrices à R_1 et R_2 coupent π suivant deux coniques osculatrices en M_0, et la considération du quatrième point commun à ces coniques fournit une construction du centre K de l'homologie (P_1, P_2). Si, pour tout couple (g_1, g_2), K coïncide avec le point caractéristique du plan (g_1, g_2), K est fixe dans l'espace, et R_1, R_2 se projettent l'une sur l'autre à partir de K avec correspondance des génératrices. L'auteur montre, au moyen d'un changement approprié de repère local, les particularités qui se présentent lorsque R_1 et R_2 sont tangentes le long de Γ_0, et termine en mettant en évidence l'opportunité de l'emploi du trièdre de Frenet de Γ_0 pour l'étude des propriétés métriques de la transformation T. *P. Vincensini.*

Gheorghiev, Gh. Étude des surfaces au voisinage d'une courbe. Bull. École Polytech. Jassy [Bul. Politehn. Gh. Asachi. Iaşi] 3, 36–47 (1948).

L'auteur étudie les voisinages des différents ordres d'une courbe Γ sur une surface S au moyen du développement de Taylor du rayon vecteur $\mathbf{R}(s, v)$ de S:

$$\mathbf{R}(s, v) = \mathbf{r}(s) + \sum_1^\infty (v^i/i!)\mathbf{r}_i(s),$$

où $\mathbf{r}(s)$ est le rayon vecteur de Γ (s étant son arc) et $\mathbf{r}_i$ une suite infinie de vecteurs définissant la surface. Il donne les formules exprimant les dérivées par rapport à s et v du vecteur $\mathbf{R}$ rapportées au trièdre de Darboux attaché à S, et en déduit les valeurs des coefficients des deux formes fondamentales de S. Il applique les formules obtenues au problème de la recherche des surfaces d'élément linéaire donné passant par une courbe donnée, retrouve simplement un certain nombre de résultats classiques concernant ce problème, et montre comment ces mêmes formules peuvent s'appliquer à la résolution du problème de Björling relatif à la détermination de la surface minima inscrite à une développable donnée le long d'une de ses courbes. Il reprend également le problème de Plateau relatif à la recherche de la calotte d'aire minima limitée par un contour fermé Γ, montre que ce problème est équivalent à celui des courbes antidérivées fermées d'une courbe fermée pour lesquelles l'image sphérique des binormales est intérieure à une demi-sphère, et établit pour un contour fermé (Γ) dérivé d'un autre contour fermé (δ) dont les binormales vérifient la condition précédente, l'identité des problèmes de Plateau et de Björling. Entre la torsion τ et l'élément d'arc ds de (δ), la courbure totale K_Γ de la surface minima de Plateau relative à (Γ) et l'élément d'arc ds_1 de (Γ), se trouve en outre établie, pour tout couple de points correspondants sur (δ) et (Γ), la relation $\tau ds = (-K_\Gamma)^{\frac{1}{2}} ds_1$. Suivant toujours la même méthode l'auteur, généralisant un théorème de Steiner, montre, qu'étant donnée une courbe gauche fermée Γ dont l'image sphérique des binormales est intérieure à une demi-sphère, le lieu géométrique des points dont les podaires par rapport à Γ déterminent une aire constante sur les surfaces minima qui les contiennent est une quadrique.

P. Vincensini (Besançon).

Constantinescu, Lucreţia. Sur les transformations infinitésimales qui conservent les directions des normales. Bull. École Polytech. Jassy [Bul. Politehn. Gh. Asachi. Iaşi] 3, 124–130 (1948).

Cette note fournit une application des méthodes vectorielles à l'étude des transformations infinitésimales d'une surface Σ conservant les directions des plans tangents. Définissant Σ vectoriellement par $\mathbf{r} = \mathbf{r}(u, v)$ et sa transformée infinitésimale Σ_1 par $\mathbf{r}_1 = \mathbf{r} + \varepsilon$ [$\varepsilon = \eta\rho(u, v)$, où η est une constante de carré négligeable], l'auteur considère d'abord le cas où la transformation est une déformation infinitésimale (conserve les arcs), et retrouve ainsi la déformation des surfaces minima, le vecteur ε étant alors $\varepsilon = \eta \int [\mathbf{N}, d\mathbf{r}]$, $\mathbf{N}$ étant le vecteur unitaire normal à Σ. Elle montre que la surface S définie par $\varepsilon_1 = \eta^{-1}\varepsilon(u, v)$ [quasi-sphère de Steiner] est aussi minima, et, au moyen de la circulation de ε le long d'une courbe fermée Γ de Σ retrouve l'expression de Schwarz pour l'aire limitée sur une surface minima Σ par un contour fermé Γ: $\Omega = \frac{1}{2}\int_\Gamma \mathbf{r} d\varepsilon = \frac{1}{2}\int_\Gamma (\mathbf{N}, \mathbf{r}, d\mathbf{r})$.

Si la transformation envisagée (conservant toujours les directions des normales) n'est plus une déformation, il existe sur Σ un réseau dont les courbes sont transformées avec égalité des arcs dans le passage de Σ à Σ_1. L'auteur recherche les conditions pour que ce réseau soit orthogonal (div $\varepsilon=0$, et la transformation conserve les aires); conjugué (le réseau est alors le réseau de base de la transformation); de courbure (on a alors une relation entre les courbures moyennes (ou totales) de Σ, Σ_1 et de la quasi-sphère S associée); asymptotique (Σ rapportée aux asymptotiques verifie alors $\{^{11}_{2}\}_{\Sigma}=\{^{11}_{2}\}_{\Sigma_1}$). *P. Vincensini* (Besançon).

Efimov, N. V. Addendum to the paper "Study of the infinitesimal deformations of certain classes of surfaces." Mat. Sbornik N.S. 22(64), 493–500 (1948). (Russian)

The author extends his previous result [Rec. Math. [Mat. Sbornik] N.S. 20(62), 27–53 (1947); these Rev. 8, 597] to deformations of a surface of zero curvature of the type of a cylindrical strip, bounded by two plane ovals: formerly the planes of the two ovals were assumed parallel.

L. C. Young (Madison, Wis.).

Aleksandrov, A. D. Foundations of the inner geometry of surfaces. Doklady Akad. Nauk SSSR (N.S.) 60, 1483–1486 (1948). (Russian)

Let M be a two-dimensional topological manifold metrized in such a way that the distance of two points equals the length of a shortest connection of the points. A geodesic triangle abc on M is a set homeomorphic to a circular disk whose boundary consists of shortest connections from a to b, from b to c, and from c to a. For any geodesic triangle abc on M let ABC denote the Euclidean triangle whose sides equal the distances of the corresponding vertices in abc. If k_1, k_2 are curves on M issuing from a point O and $x_i \varepsilon k_i$, $x_i \neq O$, and $\gamma(x_1, x_2)$ is the radian measure of the angle X_1OX_2, then $\lim\sup_{x_i \to o} \gamma(x_1, x_2)$ is called the upper angle of k_1, k_2 at O. If the limit exists, "angle" is used instead of "upper angle." The sum of the upper angles in a geodesic triangle abc, minus π, is called the excess $\epsilon(abc)$ of abc. The manifold M is said to have finitely bounded (integral) curvature if for every open region G on r with a compact closure a number $B(G)$ exists such that for any finite collection of nonoverlapping geodesic triangles $a_\nu b_\nu c_\nu$ in G the relation $\sum|\epsilon(a_\nu b_\nu c_\nu)|<B(G)$ holds.

The main theorem is this: R has finitely bounded curvature if and only if in every open region G with a compact closure the metric induced in G by the metric in M can be uniformly approximated by Riemannian metrics, whose integral curvatures are bounded (in their totality). Further results are: if M has finitely bounded curvature then angles exist for any pair of geodesic arcs (shortest connections) issuing from a point O; if the excess of every triangle in M is zero then M is locally Euclidean.

The excess yields in an obvious way a set function $\omega(T)$ on M corresponding to the integral curvature. Let $\omega(T)=\omega^+(T)-\omega^-(T)$ be the standard decomposition of $\omega(T)$ into its positive and negative parts. Let P be a geodesic polygon on M and Z: $\sum a_\nu b_\nu c_\nu$ a simplicial decomposition of P into geodesic triangles. If D_Z is the maximal diameter of the $a_\nu b_\nu c_\nu$ and $\sigma(Z)$ the sum of the areas of the triangles $A_\nu B_\nu C_\nu$, then $\sigma(P)=\lim \sigma(Z)$ exists for $D_Z \to 0$, provided M has finitely bounded curvature, and the degree of approximation of $\sigma(Z)$ to $\sigma(P)$ can be estimated by

$$-\tfrac{1}{2}\omega^-(P)d_Z^2 \leqq \sigma(P)-\sigma(Z) \leqq \tfrac{1}{2}\omega^+(P)d_Z^2.$$

No proofs are given. *H. Busemann.*

Salenius, Tauno. Über geodätische Linien in gewissen dreidimensionalen Mannigfaltigkeiten. Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. 47, 52 pp. (1948).

Let (1) $ds^2=\sum_{i,j=1}^{3} g_{ij}dx_i dx_j$ be a positive definite quadratic form, with coefficients of class C^3 in the interior of the unit sphere $\sum_{i=1}^{3}x_i^2=1$, such that (1) is invariant under a properly discontinuous group of rigid motions of the hyperbolic space defined by

$$(2) \qquad 4(dx_1^2+dx_2^2+dx_3^2)/(1-x_1^2-x_2^2-x_3^2)^2, \quad \sum_{i=1}^{3}x_i^2<1,$$

and such that the three-dimensional Riemannian manifold M obtained by identification of congruent points is closed (compact). Furthermore, let it be supposed that no geodesic has on it two mutually conjugate points. It is shown that there must exist transitive geodesics on M.

This is a generalization to three-dimensional manifolds of a result due to Morse and Hedlund [Trans. Amer. Math. Soc. 51, 362–386 (1942); these Rev. 3, 309], whose methods made strong use of the fact that the manifold under consideration was two-dimensional. The author first obtains by straightforward generalization some necessary results concerning the properties of geodesics of class A, in particular, that a class A unending geodesic is of the type of (does not wander far from) some hyperbolic line. Then the author derives the key result which states that two class A geodesic rays with the same initial point cannot be of the same type if one of them is stable in the sense of Poisson. Transitivity follows readily. As is remarked, the arguments apply equally well to analogous manifolds of more than three dimensions. *G. A. Hedlund* (New Haven, Conn.).

Castoldi, L. Sopra la derivata sostanziale di un integrale multiplo negli spazi di Riemann. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 4, 290–295 (1948).

The author considers a system of ordinary differential equations which can be regarded as representing the lines of flow of a liquid in motion in a Riemannian space V_n. With reference to this system of equations he defines the substantial manifold V_m, with coordinates x^α, as the manifold which, at the instant t, is made up of the particles which, at the initial instant t_0, occupies a certain manifold V_m^0. He calls C a substantial region of V_m when it is made up, during the motion, of the same particles of V_m. Substantial coordinates ξ^α are those which, for individual particles, remain constant during the motion. The region $\bar{C}$ of variability of ξ^α, corresponding to C, does not vary with time. A multiple integral extended over a region C can therefore be transformed into one over the constant boundary $\bar{C}$ in the ξ variables. The author uses these coordinates in order to carry out a differentiation of a multiple integral with respect to the time variable. *E. T. Davies.*

Castoldi, Luigi. Giaciture associate ad una serie di direzioni lungo una curva immersa in uno spazio a connessione affine. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. 81–82, 205–214 (1948).

Let $v^\nu(t)$ be a tangential vector of a curve C in an affine (curved) space A_n. Then (1) the space spanned by the osculating r-vector of C does not change by the transformations $'t='t(t)$, $'v^\nu=\rho(t)v^\nu$; (2) the results concerning affine arc, associated vectors and Frenet formulae established by the reviewer [Rend. Circ. Mat. Palermo 53, 365–388, 389–410 (1929)], can be immediately extended to vector fields

$w^r(t) \neq v^r(t)$ given along C or to vector fields $w^r(x)$ given in A_n. These are the main results of the paper under review.

V. Hlavatý (Bloomington, Ind.).

Wrona, W. Conditions nécessaires et suffisantes qui déterminent les espaces einsteiniens, conformément euclidiens et de courbure constante. Ann. Soc. Polon. Math. 20 (1947), 28–80 (1948).

The author derives a number of new characterizations of Einstein spaces E_n, conformally Euclidean spaces C_n and spaces of constant curvature S_n. These characteristic properties depend upon the scalar curvature of the geodesic subspace which is tangent to a given m-Bein. The author calls this quantity the scalar curvature of the m-Bein. The following results are typical. (1) A necessary and sufficient condition that V_{2n} be an E_{2n} is that the scalar curvature of any two nonsingular n-Beins which are orthogonal to each other at a point have the same scalar curvature. (2) A necessary and sufficient condition that V_n $(n>2)$ be an E_n is that all $(n-1)$-Beins at each point have the same scalar curvature. A theorem analogous to (1) is given for any C_{2n} and this, in turn, has been generalized by Haantjes [Nederl. Akad. Wetensch., Proc. 43, 91–94 (1940); these Rev. 1, 175] to apply to any C_n. The author also generalizes the classical theorem of Schur concerning spaces S_n.

A. Fialkow (Brooklyn, N. Y.).

Popovici, Constantin. Les espaces conjugués, leurs transformations infinitésimales et intégration par conditions à la limite. Ann. Soc. Polon. Math. 20 (1947), 323–334 (1948).

Two systems of X_k's (with coordinates u_j and v_k) in X_n (with coordinates x_i) are said to be conjugate if the common points satisfy the condition

$$\partial^2 x_i/\partial u_j \partial v_k = \lambda_{jk}(\partial x_i/\partial u_j) + \mu_{jk}(\partial x_i/\partial v_k)$$

[as the summation convention seems to be adopted throughout the paper, the reviewer does not see how this formula is to be interpreted]. Conjugate infinitesimal transformations are defined. In his thesis [Paris, 1908] and in his paper [same Ann. 17, 67–90 (1938)] the author has considered the case $k=1$. Here he deals with general values of n and special conjugate infinitesimal transformations. After returning temporarily to the case $n=3$ he considers several problems and limiting cases. A theorem of Königs on families of cylinders enveloping a surface is reestablished and indications are given on generalizations of higher order. There is no summary, there are no conclusions and the paper is so much abbreviated that reading it is extremely difficult.

J. A. Schouten (Epe).

Cotton, Émile. Sur un système de bivecteurs associé à un cycle. C. R. Acad. Sci. Paris 226, 1564–1566 (1948).

It is well known that to every closed curve in an E_n belongs a not necessarily simple bivector. In this paper the generalization is given for an infinitesimal curve in an X_n with an affine connection in the sense of Cartan (nonhomogeneous linear connection). The curve is given in the form $\xi^k = \xi_0^k + \lambda\varphi^k(t) + \lambda^2\psi^k(t) + \lambda^3\chi(t,\lambda)$ with an infinitesimal parameter λ and functions of t with a common period T. It is not supposed to lie on a local E_2.

J. A. Schouten (Epe).

Guillaumin, Gustave. Sur la géométrie intégrale du contour gauche. C. R. Acad. Sci. Paris 227, 32–34 (1948).

Koenigs [J. Math. Pures Appl. (4) 5, 321–342 (1889)] has shown that to every closed curve in R_3 belongs a vector (axe arcolaire) and Cotton has given the generalization for an X_n with a nonhomogeneous linear connection [cf. the preceding review]. The author draws attention to the fact that in the book "La Géométrie Intégrale du Contour Gauche" by A. Bloch and himself [to appear soon] integrals of the form

$$\int x_1^\alpha x_1^\beta x_3^\gamma \lambda^i dx_i,$$

λ^i = constant; $i=1,2,3$, along a closed curve in R_3 are considered. These integrals lead to the "characteristic affinors" of different valences of the curve. The first one is the vector of Cotton; the second one can be connected with the gravitational equations.

J. A. Schouten (Epe).

Guillaumin, Gustave. Sur diverses applications de la géométrie intégrale du contour gauche. C. R. Acad. Sci. Paris 227, 109–110 (1948).

This is a continuation of the paper reviewed above. The characteristic affinors lead to the "characteristic functional" of the curve, that can be decomposed according to the components of the characteristic affinors of different valences. In the book mentioned above many applications will be given. Generalizations to more general spaces and spaces of n dimensions are considered. Both notes are too short to give more than a vague impression.

J. A. Schouten.

Petrescu, St. Sur quelques propriétés conformes des espaces non holonomes V_n^m. Mathematica, Timişoara 23, 108–122 (1948).

Le but de l'auteur est la recherche de propriétés et invariants conformes des espaces V_n^m de Vranceanu à l'aide de résultats d'un travail antérieur [Yano et Petrescu, Disquisit. Math. Phys. 1, 191–246 (1940); these Rev. 8, 604]. Si g_{ab} $(a,b=1,\cdots,m)$ désigne le tenseur fondamental de la métrique de V_n^m, l'invariance étudiée est l'invariance par la transformation $\bar{g}_{ab}=\rho^2 g_{ab}$, où ρ est une fonction arbitraire des n coordonnées (x^i) de la variété riemannienne V_n. Des formules de transformation du tenseur d'Euler, l'auteur déduit un premier invariant conforme; d'autres proviennent de l'étude de l'invariante des classiques équations de Weingarten. L'auteur forme enfin les équations de Gauss, pour la géométrie conforme des V_n^m, et en déduit en particulier l'extension aux espaces non holonomes du tenseur de courbure conforme bien connu dû à Weyl. L'étude des cas de nullité des différents invariants ainsi mis en évidence est esquissée.

A. Lichnerowicz (Strasbourg).

Norden, A. On the inner geometry of a continuous point transformation in a plane. Doklady Akad. Nauk SSSR (N.S.) 61, 445–448 (1948). (Russian)

This is a continuation of the author's previous work on normalized surfaces in a Möbius space [Rec. Math. [Mat. Sbornik] N.S. 20(62), 263–281 (1947); Doklady Akad. Nauk SSSR (N.S.) 61, 207–210 (1948); these Rev. 9, 67; 10, 67]. The conformal plane is called normalized if to each point x corresponds the point X of intersection of the normalizing circle with the plane (the coordinates are tetracyclic). It then follows that the metric tensor must satisfy $g_{ij/k}=2l_k g_{ij}$, so that the inner geometry is that of Weyl. The author shows that the field of geodesic lines is characterized by the fact that the circles of curvature of all geodesics through x intersect again in the point X, where $x \to X$. The conformal arc for a curve is $d\sigma^2 = dk ds$, where s is Euclidean arc and k is curvature. Stationary directions are those for which the geodesics have $dk/ds=0$. It is shown that Rie-

mannian geometry is characterized by the fact that the stationary directions at each point are orthogonal. A quasi-Euclidean geometry is one admitting absolute parallelism; in such a geometry $R_{[ij]}=0$ and it is realized by taking for X the point of intersection of the circle of curvature of the pair of curves through x of an arbitrary orthogonal net. The last part of the paper deals with an isotropic correspondence, i.e., one for which dk/ds is independent of the direction of the geodesic. If this geometry is to be Riemannian the correspondence $x \to X$ must reduce to an inversion with respect to a circle. *M. S. Knebelman* (Pullman, Wash.).

Su, Buchin. On the isomorphic transformations of K-spreads in a Douglas space. II. Acad. Sinica Science Record **2**, 139–146 (1948).

In part I [same Record **2**, 11–19 (1947); these Rev. **9**, 306] the author has studied the transformations which depend upon position as well as direction, and which carry K-spreads into K-spreads. This paper deals with the Lie derivative of some of the tensors of the space and the author determines a complete set of integrability conditions for the equations of the isomorphic transformations.

E. T. Davies (Southampton).

Emde, Fritz. Divergenz und Rotor in nicht-flächennormalen Vektorfeldern. Z. Angew. Math. Mech. **25/27**, 130–131 (1947).

The author studies [in the notation of J. Spielrein, e.g., Arch. Elektrotechnik **3**, 364–369 (1915)] the geometrical meaning of divergence and curl for the field $\mathbf{t}=\mathbf{k}\times\mathbf{v}+\mathbf{k}[1+(\mathbf{k}\times\mathbf{v})^2]^{\frac{1}{2}}$, an example of a non-surface-normal field, that is nonintegrable, $\mathbf{t}\cdot\nabla\times\mathbf{v}\neq 0$, $\mathbf{t}$ a unit vector, $\mathbf{k}$ constant, $\mathbf{v}$ a differentiable field.

L. C. Hutchinson (Brooklyn, N. Y.).

Nalli, Pia. Sopra un problema relativo ai trasporti rigidi di vettori negli spazi quadrimensionali. Ann. Mat. Pura Appl. (4) **26**, 67–72 (1947).

The author discusses the following problem, which arises in the theory of rigid displacements in four dimensions. What conditions must be satisfied by a given skew-symmetric tensor A_{ij} (depending on a parameter t) in order that there exists a unit vector v_i (also depending on t), such that $A_{ij}=v_iv_j'-v_jv_i'$ $(i,j=1,2,3,4)$ for all values of t? The components of the skew-symmetric tensor are found to satisfy three necessary and sufficient conditions, of which the simplest one is the vanishing of the determinant formed by the A_{ij}. *A. J. McConnell* (Dublin).

De Donder, Th., et Van Isacker, J. Extension des identités fondamentales de la physique mathématique aux spineurs. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **34**, 317–324 (1948).

The authors consider the variational principle

$$\delta\int M dx^1\cdots dx^n=0,$$

where M is a Lagrangian function of coordinates, spinors and tensors due to exterior forces. The work of the first author in obtaining fundamental identities in the case where M contains only tensors [same Bull. Cl. Sci. (5) **33**, 108–118 (1947); these Rev. **9**, 67] is extended by considering a finite continuous group of coordinate transformations for which the spinor variation has significance, and then stating that M is a density factor in "the restrained sense." The method is illustrated for the Dirac equations, but can be used for more general density factors. It introduces the impulse-energy and kinetic moment operators directly in connection with translations and rotations. *A. Schwartz.*

Takeno, Hyôitirô. On the geometry of m-vectors. J. Sci. Hirosima Univ. Ser. A. **12**, 109–123 (1942). (Japanese)

Considering tensor analysis relating to such transformations that their coefficients have the form

$$\overset{m}{x}{}^{\alpha}_{\beta}=m!\,x^{\alpha}_{[\beta_1}\cdots x^{\alpha_m}_{\beta_m]},\quad \overset{m}{x}{}^{\alpha}_{\beta}=m!\,x^{\alpha}_{[\beta_1}\cdots x^{\alpha_m}_{\beta_m]},$$

where $\alpha_i,\beta_j=1,\cdots,n$; $a=(\alpha_1,\cdots,\alpha_m)$, $b=(\beta_1,\cdots,\beta_m)$; $a,b=1,\cdots,\nu$; $\nu={}_nC_m$, and making use of this analysis, the author discusses properties of m-vectors. In a space $\overset{m}{X}_n$ of all tensors whose coefficients of transformation have the above forms, a P-connection is defined as one in which the covariant differential of a vector A^a in $\overset{m}{X}_n$ coincides with the representation of the covariant differential of $A^{a_1\cdots a_m}$ in X_n. In particular, the author studies properties of P-connections in $\overset{m}{X}_n$ of Weyl and Riemann types and derives some theorems in X_n. *A. Kawaguchi* (Sapporo).

NUMERICAL AND GRAPHICAL METHODS

Nelšuler, L. Ya. On optimal double-entry tabulation of a function. Izvestiya Akad. Nauk SSSR. Otd. Tehn. Nauk **1948**, 1169–1191 (1948). (Russian)

This paper considers in still greater detail [see C. R. (Doklady) Acad. Sci. URSS (N.S.) **36**, 121–124 (1942); Bull. Acad. Sci. URSS. Cl. Sci. Tech. [Izvestia Akad. Nauk SSSR] **1947**, 597–608; Trav. Inst. Math. Stekloff **20**, 87–108 (1947); these Rev. **4**, 202; **9**, 104, 470] the tabulation of a function of three variables by means of double entry tables. Nine possible cases of tabulation are discussed and compared.

D. H. Lehmer (Berkeley, Calif.).

Nelšuler, L. On optimal three-fold double-entry tables for functions of three variables. Doklady Akad. Nauk SSSR (N.S.) **60**, 965–968 (1948). (Russian)

This is a résumé of several previous papers [cf. the preceding review]. *D. H. Lehmer* (Berkeley, Calif.).

Lyusternik, L. On a problem of the theory of tabulation. Uspehi Matem. Nauk (N.S.) **3**, no. 4(26), 163–166 (1948). (Russian)

This paper discusses the conditions under which a function of three variables can be tabulated by means of three double entry tables. Nelšuler [C. R. (Doklady) Acad. Sci. URSS (N.S.) **36**, 121–124 (1942); these Rev. **4**, 202] gave a condition in terms of two partial differential equations of the fourth order. The present paper inquires into the possibility of expressing this condition by a single partial differential equation of the third order and concludes that this cannot be accomplished. *D. H. Lehmer.*

***Lienard, R. Tables Fondamentales à 50 Décimales des Sommes S_n, u_n, Σ_n.** Centre de Documentation Universitaire, Paris, 1948. 54 pp.

This book originates in a discrepancy between values given by C. W. Merrifield [Proc. Roy. Soc. London **33**,

4–10 (1881), p. 7] and B. Rosser [Proc. London Math. Soc. (2) **45**, 21–44 (1938), p. 43] for the constant H in the asymptotic formula $\sum_{k=1}^{n} p_k^{-1} \sim C - H + \log\log p_n$, in which C is Euler's constant and p_k is the kth prime. The tables are all auxiliary to the purpose and give S_n, $2^{-n}S_n$, s_n, $1-s_n$, $2^{-n}s_n$, $\log S_n$, Σ_n for $n=1(1)167$, and U_n, u_n, $1-u_n$ for $n=1(1)105$. All values are to 50 decimals and odd and even values of n are tabulated separately. In these, $S_r=\sum_{n=1}^{\infty} n^{-r}$, $s_r=\sum_{n=1}^{\infty}(-1)^{n-1}n^{-r}$, $U_r=\sum_{n=0}^{\infty}(2n+1)^{-r}$, $u_r=\sum_{n=0}^{\infty}(-1)^n(2n+1)^{-r}$, $\Sigma_r=\sum_{n=1}^{\infty}p_n^{-r}$ (p_n, the nth prime). The introduction includes values of S_{2r}/π^{2r} in fractional form (depending on Bernoulli numbers) for $r=1(1)12$, and of u_{2r+1}/π^{2r+1} (depending on Euler numbers) for $r=0(1)12$. Small errors in the final digits in earlier tables are listed. Lienard's tables are more extensive than any of these, which will be found listed in section 4 of An Index of Mathematical Tables by A. Fletcher, J. C. P. Miller and L. Rosenhead [McGraw-Hill, New York, 1946; these Rev. **8**, 286]. A manuscript table of S_n to 60 decimals for $n=3(2)71$ by E. Cambi is mentioned in Math. Tables and Other Aids to Computation **2**, 139 (1946). *J. C. P. Miller* (London).

***Emde, Fritz. Jahnke-Emde. Tables of Higher Functions. Jahnke-Emde. Tafeln höherer Funktionen.** 4th ed. B. G. Teubner Verlagsgesellschaft, Leipzig, 1948. xii+300 pp.

This is a revised edition, incorporating few major changes and maintaining the general character and arrangement with bilingual text and numerous relief illustrations of the previous ones [1933, 1938]. The changes from the 1938 edition, which are listed below, represent, in the view of the reviewer, commendable improvements, and the accessions and deletions, which nearly balance in extent, seem, with perhaps one exception, felicitously chosen. Also, the annoying practice, met in many similar works, of terminating tables just one decimal digit short of the interval end, has been partly corrected.

In the third section, which deals with the error integral and related functions, the table of $\Phi(x)$ for $x=[0(.01)3.09; 5D]$ is extended in the form $e^{x^2}[1-\Phi(x)]$ for $x=[3(.01)5; 5D]$. The expressions, graphs and tables of the derivatives of $\Phi(x)$ have been dropped in favor of those of the related but more useful functions of the parabolic cylinder (Hermite functions) $\varphi_n=[\pi^{\frac{1}{2}}2^n n!]^{\frac{1}{2}}e^{\frac{1}{2}x^2}(-d/dx)e^{-x^2}$. The tables extend to higher orders than the previous ones of $\Phi^{(n)}(x)$ and cover $x=[0(.01)3.5; 5D]$, $[3.5(.1)8; 5D]$; $n=[0(1)11]$, with five significant figures maintained near zeros; they are credited to J. B. Russell [J. Math. Physics **12**, 291–297 (1933)] and checked and corrected by S. Kerridge. There is also added a short table of Laguerre functions $l_n(x)=(1/n!)e^{\frac{1}{2}x}(d/dx)^n(x^n e^{-x})$ with $x=[0(.1)1; 4D]$, $[1(.25)3; 4D]$, $[3(.5)6; 4D]$, $[6(1)14; 4D]$, $[14(2)34; 4D]$; $n=[1(1)10]$, computed by F. Tricomi [Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **76**, 288–316 (1941); these Rev. **7**, 486].

The table and graphs of the Weierstrass functions for the equianharmonic case have been omitted from the section on elliptic functions; this is no serious loss.

A table of spherical harmonics of the second kind $Q_n(x)$ due to F. Vandrey [Z. Angew. Math. Mech. **20**, 277–279 (1940); these Rev. **2**, 239] with $x=[0(.01)1; 4D)]$, $n=[0(1)7]$ has been added to the section on Legendre functions.

The largest section, which deals with cylinder and related functions, has also received the highest quota of revisions. Apart from some minor additions to the expressions given, the main accession has been a more thorough discussion including formulas and graphical illustrations on the use of semiconvergent Debye expansions for complex arguments and orders; in the real domain, the Langer approximations [Trans. Amer. Math. Soc. **33**, 23–64 (1931)] dealing with large and approximately equal arguments and order have been added. The formerly rather extended subsection on integral representations has been omitted, which seems entirely proper, as this is not a text. However, the deletion of the tables of the first order Struve functions, which in the previous edition were covered in $x=[0(.01)14.99; 4D]$, seems very regrettable in view of their increased significance in applied fields, and is, in the opinion of the reviewer, not sufficiently compensated by the inclusion of a short table of the incomplete Anger-Weber functions $\frac{1}{2}\int_0^x(\cos, \sin)[\frac{1}{2}q(\frac{1}{2}\pi t-\sin\frac{1}{2}\pi t)]dt$ for $x=[0(.1)2; 4D]$; $q=[.1(.1)1]$. A long overdue revision has been made in the correction, modernization and extension of the obsolete Dinnik tables on the functions of order $\pm\frac{1}{3}$ and imaginary argument: $J_{\frac{1}{3}}$ and $H_{\frac{1}{3}}^{1}$ are listed for $iy=[0(.1)5; 4D]$.

H. G. Baerwald (Cleveland, Ohio).

***Tables of the Bessel functions $Y_0(x)$, $Y_1(x)$, $K_0(x)$, $K_1(x)$, $0\leqq x\leqq 1$.** National Bureau of Standards. Appl. Math. Ser., no. 1, x+60 pp. (1948).

The first part gives numerical values of $Y_0(x)$ and $Y_1(x)$, to 8 decimals or 8 figures with first and second differences (except for small x) for $x=0(.0001).05(.001)1$. A ninth figure, subject to errors up to four units and consequently practically meaningless, is sometimes given. For small values $x=0(.0001).005$, the auxiliary functions $C_0(x)$, $D_0(x)$, $C_1(x)$, $D_1(x)$ are given to 8 decimals, also with first differences, where

$$Y_0(x)=C_0(x)+D_0(x)\log_{10}(x),$$
$$Y_1(x)=x^{-1}C_1(x)+D_1(x)\log_{10}(x).$$

The second part gives $K_0(x)$ and $K_1(x)$ to 7 figures, with first and second differences for $x=0(.0001).033(.001)1$. Auxiliary functions $E_0(x)$, $F_0(x)$, $E_1(x)$, $F_1(x)$ are also given to 7 decimals with first differences for $x=0(.001).03$. These are defined by

$$K_0(x)=E_0(x)+F_0(x)\log_{10}(x),$$
$$K_1(x)=x^{-1}E_1(x)+F_1(x)\log_{10}(x).$$

It is stated in the introduction that the tables were obtained by interpolation from the British Association Mathematical Tables, v. 6, Bessel Functions, Part I, Functions of Order Zero and Unity [Cambridge University Press, 1937], in spite of the unjustified aspersions cast on this source in the foreword. A slip of errata is provided: the reference to p. 42 should be ignored. *J. C. P. Miller.*

***Tables of the Bessel Functions of the First Kind of Orders Twenty-Eight Through Thirty-Nine.** By the Staff of the Computation Laboratory. Harvard University Press, Cambridge, Mass., 1948. ix+694 pp. $10.00.

The present volume contains tables of $J_n(x)$ for $n=28(1)39$ and $x=0(.01)99.99$ to 10 decimals. *A. Erdélyi.*

Zubkov, P. I. The application of a universal alternating current calculating board to the tabulation of ratios of modified Bessel functions. Izvestiya Akad. Nauk SSSR. Otd. Tehn. Nauk **1948**, 489–498 (1948). (Russian)

The instrument of calculation mentioned in the title appears to be a kind of analogue AC network analyzer using

inductances, capacities and resistances, and a certain number of amplifiers. No actual specifications are given. The paper is concerned with the application of this device to the evaluation of Bessel function ratios $zI_{n-1}(z)/I_n(z)$, where $z=i^{\frac{1}{2}}x$ (x real) and $n=\pm\frac{1}{10}$. The function is expanded in the usual way into a continued fraction which is then computed by the network (without using amplifiers). The details of this procedure do not include information as to the number of elements of the machine or the actual number of terms used in the continued fraction. There is a small table of the two functions for $x=0(.2)2$ computed to 2 decimals without the machine together with a set of corresponding readings taken from the machine. The reader is left to compute the function from these readings. The agreement is said to be within 2 per cent. *D. H. Lehmer.*

In the definition of $E_n(x)$ read e^{-xt} instead of e^{-nt}.

van de Hulst, H. C. Scattering in a planetary atmosphere. Astrophys. J. 107, 220–246 (1948).

"In this paper the problem of scattering in a planetary atmosphere, both with and without a diffusely reflecting bottom surface, is discussed. We assume that the atmospheric scattering is isotropic, with an albedo a, and that the ground surface reflects the radiation according to Lambert's law with an albedo b." [From the author's abstract.]

A discussion of the mathematical functions involved is given in an appendix. They are of three types. (1) Exponential integrals

$$E_n(x)=\int_1^\infty t^n e^{-xt}dt, \quad E_1^{(2)}(x)=\int_x^\infty t^{-1}E_1(t)dt.$$

Table I gives 6-decimal values of $\log x+\gamma$, e^{-x}, $E_1(x)$, $E_2(x)$, $E_1^{(2)}(x)$ and 5-decimal values of $2E_3(x)$. [See Fletcher, Miller and Rosenhead, An Index of Mathematical Tables, McGraw-Hill, New York, 1946, p. 207, art. 14·83; these Rev. 8, 286, for further tables of this type. There is also an extensive table by the N.B.S. Computation Laboratory.] (2) Functions

$$F_n(b,x)=\int_0^x e^{bt}E_n(t)dt$$

which can be expressed in terms of the integrals $E_n(x)$. Tables 2 and 3 give 4-decimal values of $F_n(-s,x)$, $-s=10, 5, 2, 1$, and $e^{-sx}F_n(s,x)$, $s=0, 1, 2, 5, 10$, with $n=1$ and 2. (3) Functions

$$G_{nm}(x)=\int_0^x E_n(u)E_m(u)du,$$

$$G'_{nm}(x)=\int_0^x E_n(u)E_m(x-u)du,$$

the latter of which involves the function $E_1^{(2)}(x)$ as well as $E_n(x)$. Table 4 gives 5-decimal values for $m, n=1, 2$, and 4-decimal values of two further combinations of various of these functions. All tables are for $x=0.01, 0.02, 0.05, 0.1(0.1)0.4(0.2)1(0.5)3, \infty$. *J. C. P. Miller* (London).

Hallén, Erik. Iterated sine and cosine integrals. Trans. Roy. Inst. Tech. Stockholm no. 12, 6 pp. (1947).

This gives 5-decimal values, for $x=0(0.2)14.0$, of

$$L_{11}(x)=C_{11}(x)+iS_{11}(x)=\int_0^x t^{-1}L(t)dt,$$

$$L_{21}(x)=C_{21}(x)+iS_{21}(x)=\int_0^x t^{-1}e^{it}[L(2t)-L(t)]dt,$$

in which $L(x)=C(x)+iS(x)=\int_0^x t^{-1}(1-e^{-it})dt$, where $C(x)$ and $S(x)$ are sine and cosine integrals usually denoted by $\log x+C-\mathrm{Ci}(x)$ and $\mathrm{Si}(x)$ (C is Euler's constant). Ascending and asymptotic expansions are given.

A later table of $L(x)$, denoted by $E_1(x)$ [C. J. Bouwkamp, Quart. Appl. Math. 5, 394–402 (1948); these Rev. 9, 350] gives 6-decimal values for $x=0(0.2)20$. Comparison on the two sets of values indicates a maximum disagreement of 1.1 unit in the 5th decimal, well within Hallén's claim of two units. It may be noted that Hallén's $L_{21}(x)$ and Bouwkamp's $\phi_1(x)$ are conjugate complex quantities.

J. C. P. Miller (London).

Dixon, W. J. Table of normal probabilities for intervals of various lengths and locations. Ann. Math. Statistics 19, 424–426 (1948).

The author has deposited with Brown University a table of $\int_{x-\frac{1}{2}l}^{x+\frac{1}{2}l}$ for values of $x[0(.1)5.0]$ and values of $l[0(.1)10.0]$. The values in the table may be interpreted as the probability that an observation from a normal population with unit variance will fall in an interval of length l whose midpoint is a distance x from the mean.

Extract from the paper.

Thomson, D. Halton. Approximate formulae for the percentage points of the incomplete beta function and of the χ^2 distribution. Biometrika 34, 368–372 (1947).

The four following approximations for x in

$$I_x(p,q)=\Gamma(p+q)\{\Gamma(p)\Gamma(q)\}^{-1}\int_0^x t^{p-1}(1-t)^{q-1}dt$$

are considered: (A) $x(2n,2m)\cong\exp[-\chi^2_{2m}(P)/(2n+m-1)]$, where $\chi^2_{2m}(P)$ is the P-point in the χ^2 distribution with $2m$ degrees of freedom; (B) $x(2n,2m)\cong\{(2n-1)/(2n+2m-1)\}^k$, where $k=[\chi^2_{2m}(P)]/2m$;

(C) $$x(2n,2m)\cong e^{-u}-r(1+\tfrac{7}{6}r+\tfrac{1}{6}n^{-1})u+\tfrac{1}{12}ru^2,$$

where $u=[\chi^2_{2m}(P)]/2(n+m-1)$, $r=(m-1)/2(n+m-1)$; (C) is due to Campbell [cf. Simon, An Engineers' Manual of Statistical Methods, Wiley, New York, 1941];

(D) $$x(2n,2m)\cong 2n/(2n+2me^{2z}),$$

where $z=y/(h-\lambda)^{\frac{1}{2}}+(\lambda-\frac{1}{6})(h-2n)/nA$ and where $\lambda=\frac{1}{6}(y^2+3)$, $2/h=1/n_1+1/n_2$ and y is a unit normal deviate. The results are that (1) for $n/m>6$ (A), (B) and (C) are about equally good; (2) for $6>n/m>4$, (B) and (C) are equally good; (3) for $4>n/m>1$ and $m>50$ (D) gives best results; (4) if $2>n/m$, then (B) has a maximum error of 2×10^{-3}. Other approximations [e.g., Scheffé and the reviewer, Ann. Math. Statistics 15, 217 (1944); Nair, Biometrika 35, 16–31 (1948); these Rev. 6, 9; 9, 601] are not considered.

The Wilson and Hilferty formula [Proc. Nat. Acad. Sci. U. S. A. 17, 684–688 (1931)] for approximate percentage points of chi-square is found empirically to be in error by $-Cn^{-\frac{1}{2}}$, where n is the number of degrees of freedom and C depends on the significance level chosen. For the upper 5% and 1% levels $C=+0.035$ and -0.120, respectively. Values for 11 other levels are tabulated. The adjusted approximation is within $\pm10^{-3}$ over all of Merrington's tables [Biometrika 32, 200–202 (1941); these Rev. 3, 175] for $30\leqq n\leqq100$. *J. W. Tukey* (Princeton, N. J.).

Salzer, Herbert E. Coefficients for expressing the first thirty powers in terms of the Hermite polynomials. Math. Tables and Other Aids to Computation 3, 167–169 (1948).

The numerical coefficients in the expansion of the first 30 powers of x in terms of Hermite polynomials $H_n(x)$ were computed using Glaisher's table of binomial coefficients [Messenger of Math. 47, 97–107 (1917)]. The numerical coefficients of the various powers of x in $H_n(x)$ can be derived from the given coefficients. *A. Erdélyi.*

McKeehan, L. W. Approximations to a uniform magnetic field: Roots of Legendre polynomials of even order. Rev. Sci. Instruments 19, 475 (1948).

This short note gives corrected 7-decimal values for the zeros of Legendre polynomials of orders $n=2(2)12$. The original values appeared in an earlier paper [McKeehan, same Rev. 7, 150–153 (1936)]. Many other tables have appeared giving these zeros, x, say, or giving $\frac{1}{2}x$ or $\frac{1}{2}(1+x)$, in particular, B. P. Moors [Valeur Approximative d'une Intégrale Définie, Gauthier-Villars, Paris, 1905], 16 dec. $n=2(1)10$; A. N. Lowan, N. Davids and A. Levenson [Bull. Amer. Math. Soc. 48, 739–743 (1942); errata, ibid. 49, 939 (1943); these Rev. 4, 90; 5, 110], 15 dec., $n=2(1)16$; E. R. Smith [Iowa State Coll. J. Sci. 12, 263–274 (1938)], 6 dec., $n=2(1)40$. *J. C. P. Miller* (London).

Greenwood, Robert E., and Miller, J. J. Zeros of the Hermite polynomials and weights for Gauss' mechanical quadrature formula. Bull. Amer. Math. Soc. 54, 765–769 (1948).

In connection with the quadrature formula

$$\int_{-\infty}^{\infty} e^{-x^2} f(x)dx \approx \sum_{i=1}^{n} \lambda_{i,n} f(x_{i,n}),$$

where $x_{i,n}$ are the n roots of the Hermite polynomials $H_n(x)$, the authors give a table for the weights $\lambda_{i,n}$ and the zeros $x_{i,n}$ for $n=1(1)10$, to 9 or 12 decimal places.
E. Bodewig (The Hague).

Bodewig, E. Das Hornersche Schema für komplexe Argumente. Z. Angew. Math. Mech. 28, 276–278 (1948).

The author outlines the saving in computation of the zeros of a polynomial $f(x)$ with real coefficients by Horner's method for real arguments, and for complex arguments as modified by L. Collatz [Z. Angew. Math. Mech. 20, 235–236 (1940); these Rev. 2, 61]. A short method is described for the computation of the derivative $f'(x)$ with complex arguments and of the Taylor's series expansion of $f(x)$ about x_0, a zero of $f(x)$. *E. Frank* (Chicago, Ill.).

Evangelisti, Giuseppe. Sulla soluzione approssimata delle equazioni algebriche. Mem. Accad. Sci. Ist. Bologna. Cl. Sci. Fis. (10) 3, 93–98 (1947).

After having described the principles of Gräffe's method the author gives the following device for finding by trial the true argument of a complex root from the 2^p possible arguments φ_k, where $\varphi_k=2^{-p}(\phi+2k\pi)$, $k=0, \cdots, 2^p-1$. He divides the original equation by the quadratic functions $x^2-2r\cos\varphi_m\cdot x+r^2$, where m is put successively equal to 0, $\frac{1}{4}\cdot 2^p$, $\frac{1}{2}\cdot 2^p$, $\frac{3}{4}\cdot 2^p$. The study of the fluctuations of the four linear remainders q_0x+q_1 of these divisions allows one to find the true k or at any rate to reduce the interval and to repeat the trial for a smaller interval. *E. Bodewig.*

Jahn, H. A. Improvement of an approximate set of latent roots and modal columns of a matrix by methods akin to those of classical perturbation theory. Quart. J. Mech. Appl. Math. 1, 131–144 (1948).

The method found independently by the author is in its essential part identical with that indicated without proof by Magnier [C. R. Acad. Sci. Paris 226, 464–465 (1948); these Rev. 9, 471]. He proves that the method converges quadratically (though without using this terminology). The case of coincident latent roots is reduced in the usual way. *E. Bodewig* (The Hague).

Collar, A. R. Some notes on Jahn's method for the improvement of approximate latent roots and vectors of a square matrix. Quart. J. Mech. Appl. Math. 1,145–148 (1948).

The author gives a short proof of the method described in Jahn's paper reviewed above, by means of matrix notation. *E. Bodewig* (The Hague).

Mitchell, Herbert F., Jr. Inversion of a matrix of order 38. Math. Tables and Other Aids to Computation 3, 161–166 (1948).

The author describes the technique of inverting a matrix arising in an investigation in economics by using Gauss's elimination method twice. The calculations were performed by the Aiken relay calculator. *E. Bodewig* (The Hague).

Fox, L., Huskey, H. D., and Wilkinson, J. H. Notes on the solution of algebraic linear simultaneous equations. Quart. J. Mech. Appl. Math. 1, 149–173 (1948).

The authors describe the technique of three direct methods: (1) Gauss's elimination method in three variants: the original method, the Jordan method and the pivotal condensation; (2) the method of orthogonal vectors; (3) Choleski's method. In (1) they keep one extra figure for every ten equations in order to avoid round-off errors. However, Jordan's modification requires 50% more work than the original Gauss method. The authors emphasize that (1) can be adapted to Hollerith equipment. Method (2) is rediscovered by the authors [in reality it is the old method of E. Schmidt, Rend. Circ. Mat. Palermo 25, 53–77 (1908)] and is not restricted to symmetric matrices as they believe. Method (3) is the best for a symmetric matrix and for inverting it. However, the decomposition of the system $LL'x=b$ into the sets $Ly=b$, $L'x=y$, attributed to Turing, is really the original method of Choleski himself [who, of course, did not compute the inverse of LL']. The five methods are illustrated by an example. *E. Bodewig.*

Redheffer, Raymond. Errors in simultaneous linear equations. Quart. Appl. Math. 6, 342–343 (1948).

Let the true solutions be x_i, the computed solutions x_i', so that $\sum a_{ij}x_j=b_i$, $\sum a_{ij}x_j'=b_i'$. Then the author finds the following relation between the errors of the unknowns and the errors of the right sides: $E\leqq\epsilon/\lambda^1$, where λ is the minimum characteristic root of AA'; $E^2=\sum(x_i-x_i')^2$, $\epsilon^2=\sum(b_i-b_i')^2$. From this relation it follows that $E<\epsilon(T/\nu)^{\nu/2}/|\det A|$, where $\nu=n-1$, $T=\sum\sum a_{ij}^2=$norm of A. *E. Bodewig.*

Goldberg, Edwin A. Details of the simultaneous equation solver. RCA Rev. 9, 394–405 (1948).

For a description of this machine see E. A. Goldberg and G. W. Brown, J. Appl. Phys. 19, 339–345 (1948); these Rev. 9, 535. *R. W. Hamming* (Murray Hill, N. J.).

Bickley, W. G. Difference and associated operators, with some applications. J. Math. Physics 27, 183–192 (1948).

The formal relations of 10 operators of the difference and differential calculus are collected in one table and used to derive various formulas of interpolation, and of numerical differentiation and integration. The operators concerned are those of shift (E), of forward, backward and central differences, of differentiation, of central averaging, of forward, backward and central summation and of integration. *J. Riordan* (New York, N. Y.).

Rodier, Georges. Sur un appareil permettant le calcul approché de certaines intégrales définies dans les problèmes de structure. C. R. Acad. Sci. Paris 227, 395–397 (1948).

The integral of the product of two functions is obtained by an electrical network in which two resistances are varied in accordance with the independent variable. This technique appears in several recent texts on computing machines [cf. F. J. Murray, The Theory of Mathematical Machines, King's Crown Press, New York, 1947, p. 25; these Rev. 9, 103]. *M. Goldberg* (Washington, D. C.).

Frucht, Roberto. On the numerical calculation of the perimeter of an ellipse. Math. Notae 7, 212–217 (1947). (Spanish)

The author develops approximate formulas for the perimeter P of an ellipse which do not depend on the use of tables of elliptic integrals or on polynomial approximations. In particular, for every type of ellipse, P may be approximated by $\frac{1}{2}\pi(8a+(12a^2+88b^2)^{\frac{1}{2}})$, where $2a$ and $2b$ are the lengths of the major and minor axes. *E. Frank* (Chicago, Ill.).

Levi, Beppo. On the approximate calculation of integrals. Concerning the note of Mr. Roberto Frucht. Math. Notae 7, 218–229 (1947). (Spanish)

This paper discusses methods for the procurement of bounds for the error committed by approximate formulas for integrals, and in particular for the formula for the perimeter of an ellipse given by Frucht [cf. the preceding review] for which Frucht claims (without proof) an error of less than 0.1%. *E. Frank* (Chicago, Ill.).

Lyusternik, L. A., and Ditkin, V. A. The construction of approximate formulas for the calculation of multiple integrals. Doklady Akad. Nauk SSSR (N.S.) 61, 441–444 (1948). (Russian)

In a Euclidean n-dimensional space $x=(x_1, \cdots, x_n)$ a domain Q in a k-dimensional manifold, $k \leqq n$, and a function $\varphi(x)$ on Q are given. The value of the k-fold integral $I(f)=\int_Q f(x)\varphi(x)dv$ is to be approximated by a mean value $\bar{I}(f)=\sum_{i=1}^{m} c_i f(A_i)$, c_i being constants and A_i points of Q. The approximation is said to be of order s if $I(f)=\bar{I}(f)$ whenever f is a polynomial of degree not exceeding s. Using the abbreviations $D_i=\partial/\partial\xi_i$, $(x, D)=\sum_{i=1}^{n} x_i D_i$, one has

$$\int_Q e^{(x,D)}\varphi(x)dv \cdot f(\xi)=\int_Q f(x+\xi)\varphi(x)dv,$$

$$\sum_{i=1}^{m} c_i e^{(A_i,D)} f(\xi)=\sum_{i=1}^{m} c_i f(A_i+\xi)$$

and so, to obtain an approximation of order s, the operator

$$U=\int_Q e^{(x,D)}\varphi(x)dv-\sum_{i=1}^{m} c_i e^{(A_i,D)}$$

has to transform all polynomials of degree not exceeding s into 0. A development in a power series of $D_1, \cdots, D_n$ gives immediately a system of equations from which c_i, A_i can be found.

The method is applied to

$$I(f)=\pi^{-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)e^{-x^2-y^2}dxdy.$$

Here

$$\pi^{-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{xD_1+yD_2-x^2-y^2}dxdy=e^{\Delta/4}, \quad \Delta=D_1^2+D_2^2.$$

Starting from the four points $A_1=(\rho, 0)$, $A_2=(-\rho, 0)$, $A_3=(0, \rho)$, $A_4=(0, -\rho)$ the result $c_i=\frac{1}{4}$, $\rho=1$ is obtained and a formula of order 3 is

$$I(f)=\tfrac{1}{4}[f(1, 0)+f(0, 1)+f(0, -1)+f(-1, 0)].$$

Again, starting from the points $A_1, \cdots, A_8$ at the vertices of a regular octagon inscribed in a circle around the origin with radius $\sqrt{(2-\sqrt{2})}$, and $B_1, \cdots, B_8$ in the circle with radius $\sqrt{(2+\sqrt{2})}$, four of the vertices lying on the axes [in the paper the square roots are unfortunately omitted] one finds an approximation of order 7 in

$$I(f)=\frac{\sqrt{2}+1}{16\sqrt{2}}\sum_{i=1}^{8} f(A_i)+\frac{\sqrt{2}-1}{16\sqrt{2}}\sum_{i=1}^{8} f(B_i).$$

The authors state that they have constructed formulae for integration over the area of a circle, the three-dimensional sphere and the regular polygons. *E. M. Bruins.*

Grutzmacher, Martin. A new method of representing harmonic analysis and a new mechanical curve analyser. Ministry of Aircraft Production [London], R.T.P. Translation no. 2585, 20 pp. (1943).

The original appeared in Akustische Z. 8, 49–63 (1943); these Rev. 6, 220.

Korol'kov, N. V., and Kuz'minok, G. K. An electro-integrator for the solution of ordinary linear differential equations with constant coefficients. Izvestiya Akad. Nauk SSSR. Otd. Tehn. Nauk 1948, 517–532 (1948). (Russian)

The paper describes an electrical analogue integrator constructed along the ideas of L. I. Gutenmacher [cf. C. R. (Doklady) Acad. Sci. URSS (N.S.) 47, 259–262 (1945); these Rev. 7, 221]. The circuit makes use of directed quadripole amplifiers allowing the simulation of any dynamical problem which is described by linear differential equations with constant coefficients or by a system of such equations, under various initial conditions and with arbitrary forcing functions. The speed of calculation is determined solely by the time constant of the circuits, that is, by the duration of the transients in the various parts of the circuit. These are of the order of a fraction of a second.

The equations which can be solved on the integrator are of the type

$$\sum_{k=1}^{n}(a_{ik}+b_{ik}d/dt)u_k=F_i(t), \qquad i=1, \cdots, n,$$

in which a_{ik} and b_{ik} are constants and $F_i(t)$ arbitrary functions of time. By combining the equations of the system into one by means of the usual substitutions, the electrical network can be used to solve a single differential equation with constant coefficients and arbitrary order. The circuit

consists essentially of a set of bipoles between the terminals of which there are connected resistances or capacities, and quadripoles, namely, vacuum tube amplifiers, with unsymmetrical coefficients of transmission. The circuit makes use of voltage dividers which allow the possibility of varying the influence of any of the amplifiers on all the others or upon itself. Also included in the circuit under consideration are sources of voltage. Before the problem can be solved, it must be normalized and this is achieved by first reducing all the equations to a nondimensional form and then by dividing each equation by its maximum constant coefficient so as to bring the other coefficients within certain limits, namely, $-1<a_{ik}<1$, $-1<b_{ik}<1$, $0<b_{ii}<2$. It is easier to set the coefficients on the voltage dividers if the differential equations are written in matrix form. It is important to impress the initial values of the voltages simultaneously; for, if this is not done, then after a certain length of time corresponding to the duration of the transients of the circuit the integrator will be solving a problem corresponding to initial conditions different from the ones intended.

A method is given by means of which it is possible to calculate what initial voltages must be impressed on the circuit if these cannot be impressed simultaneously. If the impressed values of the voltage are taken as $\bar{e}_{i,0}=\sum_{k=1}^{n}b_{ik}u_{k,0}$, then at $t=0$, all u_k will be equal to $u_{k,0}$ and thus the effect of a nonsimultaneous voltage application can be eliminated. The right side of the equation if periodic is given by a rotating commutator. In order to reduce the error produced by the discreteness of the segments, the commutator is connected to the integrator through a low-pass filter with a low upper limit of the pass band. Periodic functions can, of course, be given by their Fourier series if the function on the right of the equation satisfies appropriate conditions.

By repeating the process of solution periodically and by changing the sign of the right hand term of the equation each time, the voltage fed to the amplifiers will be periodic also and devoid of a constant term. It is, therefore, possible to select the frequency of repetition of the solutions so that the resulting alternating current will lie inside the transmission band of the amplifiers.

In conclusion, there is given a worked-through example of the solution of a system of six simultaneous differential equations with constant coefficients. *M. Daniloff.*

Eckart, Carl. The approximate solution of one-dimensional wave equations. Rev. Modern Physics **20**, 399–417 (1948).

The principal subject of this article is the relationship between the method of stationary phase and the Hamilton-Jacobi ray-theory. The various physical phenomena known as waves are all described by partial differential equations. In this article six cases of linear differential equations with constant coefficients are discussed. All of these have special solutions of the form $\psi(x, t)=\exp i(kx-nt)$, where the (real or complex) constants k and n are roots of the characteristic equation $W(k, n)=0$. It is found convenient to replace n by $n-i\alpha$, and to write the characteristic equation in the form $W(k, n-i\alpha)=0$ (k and n real). The single-valued solutions of this equation are $n=H_m(k)$, $\alpha=D_m(k)$ $(m=1, 2, \cdots)$. The real functions $H_m(k)$ are called the Hamiltonians of the wave equation and the functions D_m are called its logarithmic decrements. In the rest of the article the multiplicity of the Hamiltonians is ignored and the logarithmic decrements are set equal to zero. The solution of the initial value problem is given by a Fourier integra

$$\psi(x, t)=(2\pi)^{-1}\int_{-\infty}^{\infty}\varphi(k)\exp\left[i\Theta(k, x, t)\right]dx,$$

where the phase of the integrand is $\Theta=\theta(k)+kx-H(k)t$. It is shown that under certain conditions the value of the integral is determined almost entirely by the values of the integrand near the stationary points of the phase Θ.

The author discusses the group-velocity and the Hamilton-Jacobi function. A by-product is a graphical method of solving partial differential equations of the first order with constant coefficients. The rest of the article is devoted to the resolution of the waves into a spectrum, wave fronts of constant velocity, unresolved waves, the boundary value-problem and the formal theory of multiple Hamiltonians. A serious difficulty arises when there are dissipative terms, which casts doubt on the validity of the results in this case. It is shown that this difficulty results mainly from the use of the approximate method of stationary phase. The formal aspects of the problem are related to the multiplicity of Hamiltonians encountered in solving a general partial differential equation. This multiplicity was noted explicitly by Dirac [The Principles of Quantum Mechanics, 2d ed., Oxford, 1935, p. 252] and led to the prediction of the positron. The author hopes that the Hamilton-Jacobi theory can be extended so as to yield a better approximation to the solution of the wave equation, but if so, it will modify the concept of group velocity in a peculiar manner, for it will be strongly dependent on the nature of the disturbance. The article ends with some remarks about the method of steepest descent and the remainder in the method of stationary phase. These results may make it possible to overcome the difficulties noted above. However, the analytical complications will increase considerably.

S. C. van Veen (Delft).

Feenberg, Eugene. A note on perturbation theory. Physical Rev. (2) **74**, 206–208 (1948).

The linear homogeneous equations resulting when an expansion in orthogonal functions is introduced into the eigenvalue problem $H\psi=E\psi$: $(E-H_{mm})a_m=\sum_{n\neq m}H_{mn}a_n$ may be solved by the method of successive approximations and lead to the Wigner-Brillouin formulas. These involve repetitive matrix elements, i.e., a given matrix element occurs more than once in a given order. By properly regrouping these terms it is possible to derive formulas in which these repetitive terms do not occur:

$$E=H_{kk}+\sum_{m\neq k}\frac{H_{km}H_{mk}}{E-\mathcal{E}_{km}}+\sum_{m\neq k,\,n\neq km}\frac{H_{km}H_{mn}H_{nk}}{(E-\mathcal{E}_{km})(E-\mathcal{E}_{kmn})}+\cdots,$$

where

$$\mathcal{E}_{km}=H_{mm}+\sum_{r\neq km}\frac{H_{mr}H_{rm}}{E-\mathcal{E}_{kmr}}+\sum_{r\neq km,\,s\neq kmr}\frac{H_{mr}H_{rs}H_{sm}}{(E-\mathcal{E}_{kmr})(E-\mathcal{E}_{kmrs})}+\cdots,$$

$$\mathcal{E}_{kmr}=H_{rr}+\sum_{s\neq kmr}\frac{H_{rs}H_{sr}}{E-\mathcal{E}_{kmrs}}+\cdots.$$

H. Feshbach (Cambridge, Mass.).

Feshbach, Herman. On Feenberg's perturbation formula. Physical Rev. (2) **74**, 1548–1549 (1948).

Cf. the preceding review.

Kantorovič, L. V., and Gavurin, M. K. On some new processes of calculation on a tabulator, connected with the use of binary representations of numbers. Uspehi Matem. Nauk (N.S.) **3**, no. 4(26), 160–162 (1948). (Russian)

This paper explains how to convert a number to the base 2 on a punch card tabulator. This is done by successive subtractions of 2^k and discriminations. It is used in connection with a multiplication method of Akušskiĭ [cf. the following review] in which one of the factors is written to the base 2. *D. H. Lehmer* (Berkeley, Calif.).

Akušskiĭ, I. Ya. Some new methods of calculating sums of products on a tabulator. Izvestiya Akad. Nauk SSSR. Otd. Tehn. Nauk **1948**, 1193–1228 (1948). (Russian)

The author describes and compares several closely allied methods of calculating a sum $\sum_{i=1}^{k} a_i b_i$ on a punch card tabulator. The main feature is that of writing one set of numbers, say the b's, in the binary system and putting them on cards by means of "x-punching." A detailed discussion of the calculation of the sum $\sum_{i=1}^{k} a_i b_i c_i$ follows, which is applicable, however, only to $c_i \leqq 7$, but which is said to be capable of extension to larger values of c_i by carrying out the process in several stages. Some mention is made of the sum $\sum_{i=1}^{k} a_i b_i c_i d_i$, where two sets of numbers are written to the base 2, and of the sum $\sum_{i=1}^{k} a_i b_i^2$ [see also Doklady Akad. Nauk SSSR (N.S.) **59**, 1521–1524 (1948); these Rev. **9**, 622]. *D. H. Lehmer* (Berkeley, Calif.).

Akušskiĭ, I. Ya. On certain questions connected with the application of calculating-analytical machines. Trav. Inst. Math. Stekloff **20**, 39–48 (1947). (Russian)

The machines mentioned in the title are actually punch-card machines with 45 columns on a card. Besides describing the punch, the sorter, the reproducer and the tabulator, the author explains how by successive use of this much equipment to find certain linear combinations of data in which the coefficients are small positive integers. [Cf. Bull. Acad. Sci. URSS. Cl. Sci. Tech. [Izvestia Akad. Nauk SSSR] **1946**, 1081–1120; these Rev. **8**, 171.]

D. H. Lehmer (Berkeley, Calif.).

Akušskiĭ, I. Ya. Calculating-analytical machines and some of their applications to mathematical problems. Uspehi Matem. Nauk (N.S.) **2**, no. 2(18), 79–184 (1947). (Russian)

Byhovskiĭ, M. L. The new American calculating-analytical machines. Uspehi Matem. Nauk (N.S.) **2**, no. 2(18), 231–234 (1947). (Russian)

Byhovskiĭ, M. L. An electronic calculating-analytical machine (the ENIAC). Izvestiya Akad. Nauk SSSR. Otd. Tehn. Nauk **1948**, 1329–1350 (1948). (Russian)

Eckert, W. J. The IBM pluggable sequence relay calculator. Math. Tables and Other Aids to Computation **3**, 149–161 (1948).

The machine described here is a cross between the IBM sequence controlled calculator at Harvard and the standard IBM calculating punch (type 602). Only five of these machines were ever built. Since IBM seems headed toward electronic equipment the following short description should suffice. "The machine reads numbers from punched cards, performs a sequence of calculations on them by means of relay networks, and punches the results in cards. The calculating unit is the fastest relay unit that has been announced (November 1947); six-digit multiplications are performed in 0.15 second, ... The basic operations include addition, subtraction, multiplication, division, square root, and column shift. There are 36 storage and computing registers; mathematical tables with monotonic argument may be used. . . . Cards are read and punched at the rate of 100 per minute, and it is possible to read four cards simultaneously and punch a fifth at this rate."

R. W. Hamming (Murray Hill, N. J.).

Aiken, H. H., and Hopper, G. M. An automatically controlled calculating machine. Uspehi Matem. Nauk (N.S.) **3**, no. 4(26), 119–142 (1948). (Russian)

Translated from Elec. Engrg. **65**, 384–391, 449–454, 522–528 (1946); these Rev. **8**, 52.

Hartman, Philip, and Wintner, Aurel. On the effect of decimal corrections on errors of observation. Ann. Math. Statistics **19**, 389–393 (1948).

If the smallest scale division is chosen as the unit, the reading error of a scale coincides with the round-off error of absolute value less than $\frac{1}{2}$ (the decimal base is irrelevant). If t is the true value measured, ϕ the probability distribution of the pointer error $u-t$, and n (an integer) the reading, then the probability distribution of the reading error $x=u-n$, $|x| \leqq \frac{1}{2}$, is $\sum_{n=-\infty}^{\infty} \phi(x+n-t)$, and its expected value $r(t)$ is $\int_{-\infty}^{+\infty} s(x)\phi(x-t)dx$, where $s(x+1)=s(x)$ and $s(x)=x$ for $|x|<\frac{1}{2}$ [F. Zernike in Handbuch der Physik. v. 3, 1928, pp. 475–476; A. L. Lur'e, Akad. Nauk SSSR, Prikl. Mat. Meh. **11**, 489–492 (1947); these Rev. **10**, 68]. The authors show that if $\phi(x)$ is even, then

$$r(t) = -\sum_{1}^{\infty} (-1)^n (c_n/n) \sin 2\pi nt,$$

where c_n is the nth Fourier coefficient of ϕ.

The remainder of the paper is devoted to the case of Gaussian ϕ (standard deviation σ). If $q=\exp(-2\pi^2\sigma^2)$, then $r(t)=r(t,q)=-\pi^{-1}\sum_1^\infty (-1)^n n^{-1} q^{n^2} \sin 2\pi nt$. Hence, using some properties of the elliptic theta-function ϑ_2, the authors show that $r(t,q)$ is odd, analytic and of period 1 in t. In the interval $0<t<\frac{1}{2}$, $r(t,q)$ is convex in t and a decreasing function of q; so is $\max r(t,q)$ at fixed q. Lur'e [loc. cit.] proved merely that $r<1/[32\sigma^2(2\pi e)^{\frac{1}{2}}]$.

A. W. Wundheiler (Chicago, Ill.).

ASTRONOMY

***Herget, Paul. The Computation of Orbits.** Published by the author, Cincinnati, Ohio, 1948. ix+177 pp.

The book may be considered to consist of four parts. Three introductory chapters contain an exposition of essentials of the calculus of finite differences, a summary of problems in spherical astronomy, and a treatment of the problem of two bodies. The second part contains the principal methods of orbit determination, the method of Laplace, the methods of Gauss and Olbers, and the improvement of orbits. The two chapters of the third part present special perturbations and Hansen's method of general perturbations. The fourth part is an appendix containing tables.

Some of these are of interest beyond their application to methods of orbit computation: a 4-decimal critical table of Everett coefficients; a table giving coefficients for interpolating in schemes for double and single numerical integration; an "optimum interval" table which gives r^{-3} with the argument r^2. Everywhere the emphasis in the volume is upon the presentation of effective computing methods. Numerous examples are given to demonstrate the recommended technique with the use of a modern standard desk-model calculating machine. Hence the book may also serve as a manual on computational methods for those users to whom the subject of the calculations of orbits is of merely incidental interest. *D. Brouwer* (New Haven, Conn.).

Rossi, Maria. Sul problema dei due corpi di massa variabile con la legge di Eddington-Jeans. Ann. Univ. Ferrara 4, 279–287 (1943).

The author discusses a two body problem in which m, the sum of the masses of the two bodies, varies with time according to the law $dm/dt=-am^2$, where a is a positive constant. [The problem has been considered earlier by Burgatti [Rend. Sess. Accad. Sci. Ist. Bologna (N.S.) 37, 71–79 (1933)] and by Graffi [Boll. Un. Mat. Ital. 13, 95–100 (1934)].] The new results obtained are the following. (1) As t increases without limit, the number of revolutions of one body about the other increases without limit. (2) To a certain close approximation, the perihelion is stationary. *L. A. MacColl* (New York, N. Y.).

Dramba, Constantin. Les chocs triples imaginaires dans le problème rectiligne des trois corps. Acad. Roum. Bull. Sect. 29, 79–82 (1946).

Denoting the mutual distance between the masses in the rectilinear problem of three bodies by z, a_1z+u, a_2z-u, where a_1, a_2 are constants, two categories of solutions in the neighborhood of an imaginary triple collision are obtained. For solutions in the first category u, z are regular in $(t-t_0)^{2/(\lambda+2)}$, $C(t-t_0)^{2\lambda/(\lambda+2)}$, where λ is a complex constant and t_0, C denote arbitrary integration constants. The second category comprises solutions for which u, z are regular in $(t-t_0)^{\frac{1}{2}}$. *M. H. Martin* (College Park, Md.).

Cattaneo, Carlo. Attrazione newtoniana ritardata. Pont. Acad. Sci. Acta 6, 241–248 (1942).

Kopal, Zdeněk. Radial oscillations of the limiting models of polytropic gas spheres. Proc. Nat. Acad. Sci. U. S. A. 34, 377–384 (1948).

Radial oscillations of spheres of polytropic index $n=0$ and $n=5$ are considered, in each case by transforming the differential equation of oscillation into a hypergeometric form. For $n=5$, the model is transformed homologously into one of finite mass and radius and infinite central condensation: such a model is shown to have no finite periods of oscillation. The results are interpreted as confirming those of Sen [Proc. Nat. Acad. Sci. India. Sect. A. 12, 99–107 (1942); these Rev. 8, 60]. *T. G. Cowling* (Leeds).

Ghosh, N. L. A note on the equilibrium of fluid matter in a steady differential rotation. Bull. Calcutta Math. Soc. 39, 131–138 (1947).

Rotation is considered of a gravitating fluid mass when the gravitational potential is a function of the density alone. Two special cases are considered, in each of which the gravitational field is due to a heavy solid spherical core. In the first, the fluid density is taken to vary inversely as the cube of the distance from the core's centre, and the fluid is spheroidal: in the second, the pressure-law assumed corresponds to fluid in a rotating anchor ring. *T. G. Cowling* (Leeds).

✝Letestu, Serge. Sur les Mouvements Internes d'un Astre Fluide avec Viscosité. Thesis, University of Geneva, 1942. 47 pp.

First there is considered the decay of the relative motion of a finite number of concentric solid shells, each touching that next in size, and exerting on it a frictional force proportional to the difference of the two's angular velocity. Next the results obtained in the limit as the number of shells is increased indefinitely are shown to be identical with those for a slowly rotating viscous gravitating liquid sphere, whose angular velocity depends only on the distance from the centre. The problem of the decay of annular motions in the liquid sphere is also solved. Lastly, Maclaurin's results on rotating spheroids are rederived. *T. G. Cowling.*

Pecker, Jean-Claude. Sur une méthode variationnelle de recherche des solutions approchées de l'équation de transfert. C. R. Acad. Sci. Paris 226, 1889–1891 (1948).

The principle $\int_0^\infty[F(t)-F]^2dt=$ minimum which has recently been used by V. Kourganoff [same C. R. 225, 491–493 (1947); these Rev. 9, 190] for the solution of the equation of transfer of an isotropically scattering atmosphere is generalized in the form $\sigma(\tau)=\int_0^\tau[F(t)-F]^2dt=$ minimum. The "best" linear representations for the source function B, valid in the interval $(0, \tau)$, are obtained in this manner. *S. Chandrasekhar* (Williams Bay, Wis.).

Pecker, Jean-Claude. Sur le rapport entre la température de surface et la température effective des étoiles en équilibre radiatif. C. R. Acad. Sci. Paris 226, 1961–1962 (1948).

The minimal principle recently suggested by V. Kourganoff [same C. R. 225, 1124–1126 (1947); these Rev. 9, 310] for the solution of the problem of radiative equilibrium of a nongray atmosphere is formulated in a slightly different form and the conclusion is reached that the relation between the effective and the boundary temperatures is independent of the equation of hydrostatic equilibrium. *S. Chandrasekhar* (Williams Bay, Wis.).

RELATIVITY

Lampariello, Giovanni. Il problema degli n corpi nella relatività generale. Pont. Acad. Sci. Comment. 6, 813–863 (1942).

The author treats the n-body problem in general relativity as a limiting case of the motion of a continuous medium whose parts do not interact (so that we may neglect the internal stress). Starting from the well-known field equations for such a medium, he proceeds to linearize the equations by the usual approximation methods. The solution of the approximate equations and its interpretation are to be treated in a further paper. *A. J. McConnell.*

Einbinder, Harvey. Generalized virial theorems. Physical Rev. (2) 74, 803–805 (1948).

Simple proof of the virial theorem for the case of classical relativistic mechanics. *L. Tisza* (Cambridge, Mass.).

Flint, H. T. The quantization of space and time. Physical Rev. (2) **74**, 209–210 (1948).

If the physical world is described by a five-dimensional continuum in which x_1, x_2, x_3, x_4 are space-time coordinates and x_5 occurs only in the factor $\exp(2\pi i x_5/l_0)$, where l_0 is a fundamental length interpreted as the Compton wavelength for a particle of rest-mass m_0, it is meaningless in describing the path of the particle to attribute to x_5 any value less than l_0. Snyder's variables η_μ ($\mu=1, 2, 3, 4, 5$) and coordinate operators x_m [same Rev. (2) **71**, 38–41 (1947); these Rev. **8**, 412] are introduced assuming that η_5 occurs only in the factor $\exp(2\pi i\eta_5/l_0)$, and may be assigned the value l_0 in any equation of physical significance. The difference between the observed value of x_m and the average value of η_m is such that there is no physical significance in associating with the particle a proper time interval less than h/m_0c^2. *C. Strachan* (Aberdeen).

Blanuša, Danilo. Quelle est la géométrie sur une plaque en rotation? Hrvatsko Prirodoslovno Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. **1**, 97–111 (1946). (Croatian. French summary)

The Galilean line-element, with cylindrical spatial coordinates r, ϕ, z, is $ds^2=dr^2+r^2d\phi^2+dz^2-c^2dt^2$. A system rotating with angular velocity ω has coordinates $(\rho, \psi, \zeta, \tau)$ connected with the former system by the equations $r=\rho$, $\phi=\psi+\omega\tau$, $z=\zeta$, $t=\tau$, whence

$$ds^2=g_{ij}dx^idx^j\equiv d\rho^2+\rho^2d\psi^2+d\zeta^2+2\rho^2\omega d\psi d\tau+(\rho^2\omega^2-c^2)d\tau^2.$$

The author, having reestablished the known formula $d\sigma^2=\sum_1^3(g_{\mu\nu}-g_{44}^{-1}g_{\mu 4}g_{\nu 4})dx^\mu dx^\nu$ for the spatial line-element, uses it to obtain $d\sigma^2=d\rho^2+\rho^2(1-\rho^2\omega^2/c^2)^{-1}d\psi^2$ for the line-element of the disc itself. The circumference of a disc of radius ρ is therefore greater than $2\pi\rho$. For small ρ the geometry of the disc is approximately Lobachevskian. The paper concludes with a few remarks on the physical consequences of this result. *H. S. Ruse* (Leeds).

Lichnerowicz, André, et Fourès-Bruhat, Yvonne. Théorème global sur les ds^2 extérieurs généraux d'Einstein. C. R. Acad. Sci. Paris **226**, 2119–2120 (1948).

Suppose we have an exterior ds^2, everywhere regular, for which the Einstein equations $R_{\alpha\beta}=0$ are satisfied, and also a ds^2 which (i) satisfies the interior equations $R_{\alpha\beta}=\chi(\rho u_\alpha u_\beta-pg_{\alpha\beta})$ inside a given world-tube of which the bounding hypersurface is generated by time-curves; (ii) coincides with the former ds^2 outside the tube; and (iii) agrees with it on the boundary. Then the authors show that, if the density of matter is small, the assumption of global regularity leads to a contradiction. Therefore, since an interior ds^2 is always everywhere regular in the domain covered by it, they conclude that the tube necessarily contains singularities of the external field. *H. S. Ruse.*

Papapetrou, A. The question of non-singular solutions in the generalized theory of gravitation. Physical Rev. (2) **73**, 1105–1108 (1948).

The theorem that the field equations of the relativistic theory of gravitation do not admit a nonsingular static solution representing a field of nonvanishing total mass is shown to hold in the generalized theory of gravitation with nonsymmetric g_{ik}. *M. Wyman* (Edmonton, Alta.).

Papapetrou, A. A static solution of the equations of the gravitational field for an arbitrary charge-distribution. Proc. Roy. Irish Acad. Sect. A. **51**, 191–204 (1947).

The author shows that the gravitational potentials in the line element

$$ds^2=\frac{dt^2}{(1+\epsilon\phi)^2}-(1+\epsilon\phi)^2(dx^2+dy^2+dz^2)$$

will be solutions of the field equations if the function ϕ satisfies Laplace's equation $\nabla^2\phi=0$. This result is then extended by introducing Poisson's equation to replace that of Laplace. By means of the above results the author is able to discuss the behavior of singularities in a field of point changes. *M. Wyman* (Edmonton, Alta.).

Papapetrou, A. Einstein's theory of gravitation and flat space. Proc. Roy. Irish Acad. Sect. A. **52**, 11–23 (1948).

The possibility of formulating a theory of gravitation within the framework of the flat space of special relativity has been considered by various people during the past few years. In the present paper the author discusses the significance of the law of conservation of angular momentum for such a formulation. The asymptotic values of the gravitational potentials are obtained for a system having a given mass and a given angular momentum. *M. Wyman.*

Einstein, Albert. A generalized theory of gravitation. Rev. Modern Physics **20**, 35–39 (1948).

In previous papers the author has formulated a generalized theory of gravitation in which the gravitational potentials constitute a Hermitian tensor. The present paper gives a new presentation of this theory. *M. Wyman.*

Jordan, Pascual, und Müller, Claus. Über die Feldgleichungen der Gravitation bei variabler "Gravitationskonstante." Z. Naturforschung **2a**, 1–2 (1947).

In P. Jordan's five-dimensional projective relativity theory the so-called "constant of gravitation" χ ($=8\pi G/c^2$) is allowed to vary. The components $g_{\mu\nu}$ ($\lambda, \mu, \nu=0, 1, \cdots, 4$) of the five-dimensional fundamental metric tensor are homogeneous functions of degree -2 in the projective coordinates X^λ, and χ is related to the scalar $J=g_{\mu\nu}X^\mu X^\nu$ by $J=2\chi/c^2$. World coordinates x^k ($i, j, k=1, \cdots, 4$) are functions of the X^λ, and, writing g_ν^k for $\partial x^k/\partial X^\nu$, the contravariant fundamental tensor in four dimensions is given by $g^{ik}=g_\mu^ig_\nu^kg^{\mu\nu}$. The electromagnetic tensor is identified with

$$F^{ik}=J^{-1}g_\mu^ig_\nu^k(\partial X_\mu/\partial X^\nu-\partial X_\nu/\partial X^\mu)g^{\mu\nu}g^{\nu\nu}.$$

The equations resulting from the vanishing of the five-dimensional Ricci tensor $R_{\mu\nu}$ are examined.

A. G. Walker (Sheffield).

Ludwig, Günther. Der Zusammenhang zwischen den Variationsprinzipien der projektiven und der vierdimensionalen Relativitätstheorie. Z. Naturforschung **2a**, 3–5 (1947).

This is concerned with variational principles in P. Jordan's five-dimensional projective relativity theory [see the preceding review]. It is shown that the system of Euler equations arising out of a five-dimensional variational equation can be similarly derived from a four-dimensional variational equation. *A. G. Walker* (Sheffield).

Hély, Jean. La théorie de Birkhoff. Revue Sci. **86**, 115–120 (1948).

An exposition of G. D. Birkhoff's theory of relativity.

Gião, Antonio. Sur les transformations de Lorentz internes et externes et le vent d'éther. C. R. Acad. Sci. Paris 226, 2051–2053 (1948).

Thiry, Yves. Sur la régularité des champs gravitationnel et électromagnétique dans les théories unitaires. C. R. Acad. Sci. Paris 226, 1881–1882 (1948).

The author establishes the theorem that if the gravitational and electromagnetic fields of a stationary space-time are such that at infinity the space-time is Euclidean and the electromagnetic field is zero then these fields cannot be everywhere regular without reducing identically to zero. This result is shown for the five-dimensional field theory of Klein and Kaluza and the generalization of it investigated by the author [same vol., 216–218 (1948); these Rev. 9, 386]. *H. C. Corben* (Pittsburgh, Pa.).

Takeno, Hyôitirô. Equations characterizing various Riemannian spaces treated in cosmologies. J. Sci. Hirosima Univ. Ser. A. 12, 125–136 (1942). (Japanese)

In the first part it is shown that spherically symmetrical solutions of the equations $\nabla_h K_{ijlm}=0$ must be one of the following: for $B\neq$constant,

$$ds_1^2=-\frac{dr^2}{1-k^2r^2}-r^2d\theta^2-r^2\sin^2\theta d\phi^2+(1-k^2r^2)dt^2$$

(type of de Sitter),

$$ds_2^2=-dr^2-r^2d\theta^2-r^2\sin^2\theta d\phi^2+dt^2$$

(type of Minkowski),

$$ds_3^2=-\frac{dr^2}{1-r^2/k^2}-r^2d\theta^2-r^2\sin^2\theta d\phi^2+dt^2$$

(type of Einstein); for $B=$constant,

$$ds_4^2=(1/pr^2)(-dr^2+dt^2)-B(d\theta^2+\sin^2\theta d\phi^2),$$

$$ds_5^2=-dr^2+dt^2-B(d\theta^2+\sin^2\theta d\phi^2);$$

and that spherically symmetrical solutions of $\nabla_h K_{ij}=0$ must be one of the above or of the following:

$$ds_6^2=-\frac{dr^2}{1-m/r}-r^2d\theta^2-r^2\sin^2\theta d\phi^2+(1-m/r)dt^2$$

(type of Schwarzschild),

$$ds_7^2=-\frac{dr^2}{1-k^2r^2-m/r}-r^2d\theta^2-r^2\sin^2\theta d\phi^2+(1-k^2r^2-m/r)dt^2.$$

In the second part the author classifies these space-times and similar ones according to properties of the eigenvalues of a matrix $[K_a^b]$ $(a, b=1, \cdots, 6)$ formed from the components of $K_{ij}{}^{lm}$. *A. Kawaguchi* (Sapporo).

Takeno, Hyôitirô. On the mathematical representation of light in cosmology in terms of wave geometry. J. Sci. Hirosima Univ. Ser. A. 13, 163–172 (1944). (Japanese)

In wave geometry the state function ψ in cosmology is derived from the general solution of the fundamental equation $\nabla_i\psi=\frac{1}{2}k\gamma_i\psi$, imposing the condition $(u^r/u_l)_{r=0}=0$. The state function ψ of light is defined by

$$[\text{general solution } \psi]+[(v^l\gamma_l\psi)_P=0]=[\psi \text{ of light}],$$

where light starts from the point $P(x)$ in the direction v^i (of course $v^lv_l=0$) and the author investigates some properties of light defined in this way. The author remarks finally that N, u_{ijkl}; u^i, u_{ijkl}, ψ; u_{ij}, u_{ijk}; u_{ik}, u_{ijk}; M, u_{ijkl} are solutions of the equation $g^{ij}\nabla_i\nabla_jF=\mu k^2F$ corresponding to integral values 0, 1, 2, 3, 4 of μ, respectively. *A. Kawaguchi* (Sapporo).

Takeno, Hyôitirô. Comparison between the cosmology in terms of wave geometry and the kinematical cosmology from the standpoint of the momentum-density vectors. J. Sci. Hirosima Univ. Ser. A. 13, 327–330 (1944). (Japanese)

It is shown that the momentum-density vector u^i in wave geometry is composed of $\bar{u}^i$ of Milne's type obtained by McVittie and $\overline{\overline{u}}^i$ of Einstein's type. It is natural to interpret $\overline{\overline{u}}^i$ as the principal part and $\bar{u}^i$ as an addition. There are some considerations about the difference between the cosmology in terms of wave geometry and that of Milne. *A. Kawaguchi* (Sapporo).

Takeno, Hyôitirô. On M, N, u^i, $\cdots$, etc. in the cosmology in terms of wave geometry. J. Sci. Hirosima Univ. Ser. A. 13, 331–337 (1944). (Japanese)

The vector $u^i=\psi^\dagger A\gamma^i\psi$ plays an important rôle in the cosmology in terms of wave geometry. Except u^i, many quantities can be formed from γ^i and ψ, e.g., $M=\psi^\dagger A\psi$, $N=\psi^\dagger A\gamma_5\psi$, $u^{ij}=\psi^\dagger A\gamma^i\gamma^j\psi$, $u_{ij}=\psi^\dagger A\gamma_{(i}\gamma_{j)}\psi$, etc. However, their physical meanings are not yet sufficiently clear. In this paper their concrete forms are obtained in various coordinate systems and relations of 16 constants contained in these quantities are brought to light. *A. Kawaguchi* (Sapporo).

Sibata, Takasi. Improvement and supplement of the wave geometrical theory of the cosmos and nebulae. J. Sci. Hirosima Univ. Ser. A. 12, 137–140 (1942). (Japanese)

Criticizing the theory of the cosmos in wave geometry from the viewpoint of homogeneity for observation systems and of the relation between the momentum-density vector and geodesic lines, the author feels a lack of consistency in this theory. It is shown that the theory can be improved from the standpoint of "homogeneity of space-time for observation systems" proposed by the author [same J. 11, 231–243 (1941); these Rev. 9, 627]. In the fundamental equations of ψ in the theory of nebulae [T. Iwatsuki and T. Sibata, same J. 11, 47–88 (1941); these Rev. 3, 63] the relations $A_{(ij)}=A^k_{[ij]}=0$ are derived from physical considerations. The author shows that these relations can be obtained from the conditions of integrability of the fundamental equation, if $A_i=A_{ik}=0$ holds good. *A. Kawaguchi* (Sapporo).

Sibata, Takasi. On the classification of spherically symmetric matrices. J. Sci. Hirosima Univ. Ser. A. 12, 255–267 (1943). (Japanese)

In this paper the definition of homogeneity suggested by the author [see the preceding review] is investigated for microscopic phenomena. First it is shown that ψ must be transformed into $'\psi=S^{-1}\psi$, in order that the linearized operator $dx^l\gamma_l$ have some eigenvalues for such transformations of coordinates $'x^i='x^i(x)$ that $'g_{ij}=g_{ij}(x')$, where S is a matrix defined by $'\gamma_i=S\gamma_i(x')S^{-1}$. From this theorem it follows that homogeneity of the operator requires that ψ must be a spinor. Next, considering matrices Σ_i in the fundamental equations $\partial\psi/\partial x^i-\Gamma_i\psi=\Sigma_i\psi$, the author shows that spherical symmetry of Σ_i should be determined by that of

the totality of a system of matrices

$$\frac{\partial x^k}{\partial' x^i}\Sigma_k(x;a,b,c,\cdots)={}'\Sigma_i=S\Sigma_i(x';a',b',c',\cdots)S^{-1},$$

where $a, b, c, \cdots$ be parameters representing a system of matrices. Then the matrices are classified according to the number of parameters and it is proved that the number of parameters must be essentially equal to 2 at most. Finally the author suggests the possibility of a development of hydrogen atom theory. *A. Kawaguchi* (Sapporo).

MECHANICS

Blaschke, Wilhelm. Über Integrale in der Kinematik. Arch. Math. 1, 18–22 (1948).

J. Steiner [Gesammelte Werke, v. 2, Berlin, 1882, pp. 99–159 = J. Reine Angew. Math. 21, 33–63, 101–133 (1840)] established a formula for the area enclosed by the trajectory of a point in a periodic coplanar motion of a plane rigid system. The author extends Steiner's result to the rotation of a rigid body about a fixed point and to the general motion of a rigid body in three-space. *W. Prager.*

Zwikker, C. Vectorial theory of gear wheel tooth profiles. Appl. Sci. Research A. 1, 139–150 (1948).

The complex variable is applied to gear teeth in the determination of tooth shapes, the path of the point of contact and the slip. The most common tooth profile, the involute, is expressed by $z=a(1-j\omega)\exp(j\omega)$. The epicycloidal profiles, which are sometimes used, are given by $z=(n+1)a\exp(j\omega)$, while the trochoids traced by points on the moving gears are given by $z=a_1\exp(jn_1\omega)+a_2\exp(jn_2\omega)$. *M. Goldberg* (Washington, D. C.).

Poritsky, H., and Dudley, D. W. Conjugate action of involute helical gears with parallel or inclined axes. Quart. Appl. Math. 6, 193–214 (1948).

Helical gears are used in modern applications to give smoother action by reducing the cogging action of the more familiar spur gears. The working surface of an involute helical gear is generated by a plane involute which moves with a uniform screw motion along the axis normal to the base circle at its center. The equations of this surface and some of its geometry are derived. It is shown to be the developable surface which is the tangent surface of a helix. The spherical indicatrix of the straight lines on this surface is a small circle. Two gears can operate together if the spherical indicatrix of one gear is tangent to, or intersects, the spherical indicatrix of the other. Several theorems are derived giving limitations on the parameters of mating pairs, the types of contact and the motions of the points of contact. Practical applications include the effects of the number of teeth, of displacements between axes and of improper angular relations. *M. Goldberg.*

Melchior, Paul. Sur la dynamique des solides. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 34, 445–448 (1948).

A method is described for obtaining the Euler equations for the motion of a rigid body. The method is based on an isoperimetric form of Hamilton's principle, the side conditions expressing the fact that the body moves rigidly. *D. C. Lewis* (Baltimore, Md.).

Aczél, John, and Fenyö, Stephen. On fields of forces in which centres of gravity can be defined. Hungarica Acta Math. 1, no. 3, 53–60 (1948).

The authors seek fields in which the field effect is rigorously equal to that on the total mass concentrated at the center of gravity, with an added condition that, for points on a straight line, the center is on the line. They find such fields necessarily parallel or central, with the magnitude of the forces in general proportional to the distance from a line in the plane, or from a plane in space. The special cases of constant parallel forces or central forces proportional to the distance appear. *P. Franklin* (Cambridge, Mass.).

Pinney, Edmund. Aerodynamic oscillations in suspension bridges. J. Appl. Mech. 15, 151–159 (1948).

The author describes the oscillations of a suspension bridge having two towers, two side-spans and one mid-span. He takes into consideration the bending and torsion of the road-bed, the bending of the towers and the twist of the towers round a vertical axis. The flutter-mode is assumed to be a linear combination of one bending mode and one torsional mode of the road-bed in vacuum. The critical airspeed is computed as a function of the structural damping by the use of the Theodorsen flap-plate air-forces for two-dimensional flow [Tech. Rep. Nat. Adv. Comm. Aeronaut., no. 496 (1934)]. *W. H. Muller* (Amsterdam).

García, Godofredo. Reduction of the pendular equations to V. Volterra's equation of the second kind. Pont. Acad. Sci. Acta 4, 147–152 (1940). (Spanish)

Using simple and entirely commonplace methods, the author reduces the system of differential equations of motion and initial conditions of a not very clearly stated dynamical problem to a system of equations which consists of one Volterra integral equation of the second kind and certain associated explicit formulae. The equations are put in two forms, adapted for use in the cases in which the values of a certain function are large and small, respectively. *L. A. MacColl* (New York, N. Y.).

Stange, K. Über die Bewegung des stabilen schweren symmetrischen Kreisels bei kleinen Neigungswinkeln seiner Achse. Ing.-Arch. 16, 121–134 (1947).

This paper concerns itself with the small oscillations of a stable gyroscope. In describing such motions the position of the gyroscope's axis may be specified either by means of the first two Eulerian angles (α, ψ) or by means of the first two direction cosines (x, y); α oscillates about its mean value with the nutation frequency ω_N and ψ increases with the mean angular velocity ω_P. It then turns out that the linearized equations for x and y involve only the frequencies ω_P and $\omega_P+\omega_N$. *D. C. Lewis* (Baltimore, Md.).

Teofilato, Pietro. Caratteri giroscopici derivanti da valori iniziali sufficientemente grandi delle velocità ignorabili. Pont. Acad. Sci. Acta 8, 157–175 (1944).

This paper is concerned with certain properties of a general conservative holonomic dynamical system with N degrees of freedom, having n ignorable coordinates $q_1, \cdots, q_n$ and $N-n$ nonignorable coordinates $Q_1, \cdots, Q_{N-n}$. If T denotes the kinetic energy of the system, the partial derivatives $\partial T/\partial\dot{q}_i$ are constants. The author writes $\partial T/\partial\dot{q}_i=k\gamma_i$, the positive parameter k serving as a convenient measure of the magnitude of the initial values of the $\dot{q}_i$. It is shown that

there exist $Nn-n^2$ functions $F_j(Q_1, \cdots, Q_{N-n}, \dot{Q}_1, \cdots, \dot{Q}_{N-n})$ which are linear and homogeneous in the $\dot{Q}_i$, and which remain small in absolute value throughout all time, provided that the initial values of the F's are numerically small, and provided that the initial values of the $\dot{q}_i$ are such as to make k large. From this it follows that, under the same conditions, the $\dot{Q}_i$ remain numerically small throughout all time. Thus the system possesses what may be regarded as gyroscopic properties, in a general sense of the term. In the final section of the paper the general theory is used in a short discussion of a system containing several rotating bodies. *L. A. MacColl* (New York, N. Y.).

Mandel, Jean. Sur l'invariance des équations de la mécanique dans un changement du système de référence. C. R. Acad. Sci. Paris **227**, 326–328 (1948).

This note is devoted to some general considerations concerning the invariance of dynamical equations under transformations of the system of reference. Since the author does not specify exactly what transformations he is discussing, his reasoning and conclusions remain somewhat vague. The reviewer has not found any results in the note that can be regarded as essentially new.

L. A. MacColl (New York, N. Y.).

Hydrodynamics, Aerodynamics, Acoustics

Predvoditelev, A. S. On a molecular-kinetic derivation of the equations of hydrodynamics. Izvestiya Akad. Nauk SSSR. Otd. Tehn. Nauk **1948**, 545–560 (1948). (Russian)

L'auteur reprend les fondements de la théorie cinétique de Maxwell. Après avoir établi les équations de mouvement d'un fluide visqueux, il essaye de former les équations du mouvement en rejetant l'hypothèse de Maxwell: la vitesse de transport de deux molécules qui se choquent sont égales. On trouve de cette façon des équations de mouvement généralisants celles de Navier-Stokes. L'auteur applique ces résultats aux courants gazeux à deux dimensions de Tchapliguine. *M. Kiveliovitch* (Paris).

Frenkel', Ya. I. On the behavior of liquid drops on the surface of a solid body. I. A rolling drop on a sloping surface. Akad. Nauk SSSR. Zhurnal Eksper. Teoret. Fiz. **18**, 659–667 (1948). (Russian)

The author considers the problem of a two-dimensional liquid drop on a sloping surface, taking account of gravitational and capillary forces. He finds that, if α^* is the critical angle of slope of the surface beyond which the drop begins to roll, the following relation is satisfied: $2\sigma = gm \sin \alpha^*$, where σ is the surface tension of the gas-liquid interface, m is the mass (per unit length) of the drop, and g is the gravitational acceleration. The author was not able to find analytically the angles of contact of the drop at its forward and rear boundaries and, on the basis of experimental evidence, assumes them equal to $\theta_0 \pm \Delta\theta$, where θ_0 is the angle of contact for the drop on a horizontal surface. With this assumption he is able to find the shape of the drop. The author also discusses in a general way some dynamic and hysteresis problems associated with drops and bubbles which he intends to study later. *J. V. Wehausen.*

Jacob, Caïus. Sur le mouvement fluide bidimensionnel produit par la rotation de deux lames en prolongement. Acad. Roum. Bull. Sect. Sci. **25**, 511–514 (1943).

Let A_1A_2 and A_3A_4 be two laminas such that A_1, A_2, A_3, A_4 lie on a straight line L. This paper considers the plane irrotational motion (without circulation) of an ideal incompressible fluid produced by the rotation of the laminas about a point O on L between them. Explicit formulas are given for the complex velocity potential of the motion, and for the resultant force and moment on the laminas. E.g., the resultant moment about O is $M = -\frac{1}{8}\pi\rho\dot{\omega}(t)(q^2-4s)$, where ρ is the density of the fluid, $\omega(t)$ is the angular velocity of rotation, and, if $x_i = \overline{OA}_i$, $q = \sum_{i<j} x_i x_j$, $s = x_1x_2x_3x_4$.

D. Gilbarg (Bloomington, Ind.).

Bondi, H., and Lyttleton, R. A. On the dynamical theory of the rotation of the earth. I. The secular retardation of the core. Proc. Cambridge Philos. Soc. **44**, 345–359 (1948).

The authors consider only the tidal friction couple acting on the outer shell and the motion of a viscous core differing from uniform rigid body rotation. The liquid is supposed to be contained in a sphere of radius a rotating with angular velocity $n+kt$ (n = constant, k = constant <0, $|k| \ll n^2$). Two equations of fourth order then give the stream function ψ in a meridian plane and the velocity about the axis $w = \Omega/r$. These follow from the general equations for any axially symmetrical motion. It is assumed that in the case of the earth ψ and $\Omega = r^2n + kr^2t + \chi(r, z)$ are small quantities (r, z are cylindrical coordinates). Then these quantities will satisfy two linear equations, which in the case of an established motion become

$$\nu\left(\frac{\partial^2}{\partial z^2}+\frac{\partial^2}{\partial r^2}-\frac{1}{r}\frac{\partial}{\partial r}\right)\chi + 2n\frac{\partial\psi}{\partial r} = kr^2,$$

$$\nu\left(\frac{\partial^2}{\partial z^2}+\frac{\partial^2}{\partial r^2}-\frac{1}{r}\frac{\partial}{\partial r}\right)^2\psi - 2n\frac{\partial\chi}{\partial z} = 0.$$

The solutions are given under the boundary conditions $R = a$ (R, θ being polar coordinates), $\psi = \chi = 0$, $\partial\psi/\partial R = 0$, for an interior region $a - R \gg l$ and for a boundary layer of depth l, where the viscosity is of importance. The core, on the whole, always rotates slightly faster than the shell. The circulation in meridian planes consists of a flow towards the equatorial plane in the interior and in the opposite direction in the boundary layer. The relative motion of a liquid core will not have any considerable influence on the action of the tidal friction couple, as if the earth were regarded as a rigid body. The corresponding two-dimensional problem is also treated. *W. Jardetzky* (Graz).

Hicks, Bruce L. Diabatic flow of a compressible fluid. Quart. Appl. Math. **6**, 221–237 (1948).

This paper is a general study of the branch of aerodynamics which deals with the steady but nonadiabatic ["diabatic"] flow of an ideal frictionless gas. Different forms of the equations of motion in both three and two dimensions are given, with adiabatic flows as special case, and some principal characteristics of such type of flows are also discussed. No specific problems are solved.

Y. H. Kuo (Ithaca, N. Y.).

Stewart, H. J. The lost solutions in axially symmetric irrotational compressible fluid flow. Quart. Appl. Math. 6, 334–337 (1948).

It is shown that an axially symmetric potential gas-flow whose hodograph (in a meridian plane) is one-dimensional is either a flow due to a two-dimensional line source superimposed on a uniform flow, or the Taylor-MacColl flow past a cone. *L. Bers* (Syracuse, N. Y.).

Zerner, Frédéric. Sur les jets supersoniques plans. C. R. Acad. Sci. Paris 227, 175–177 (1948).

The problem of supersonic plane jets is generally solved numerically or graphically with the aid of the method of characteristics. The author proposes to use the Legendre transformation and construct the solution in the hodograph plane. The condition on the transformed potential χ at the free surface is first derived and then χ is constructed by using either the method of successive approximations or the method of Riemann. The second method reduces the problem to that of solving an integrodifferential equation. The author does not mention the possibility of singularities in the Legendre transformation, which causes the breakdown of solution so constructed and the appearance of shocks. *H. S. Tsien* (Cambridge, Mass.).

Frankl', F. I. The flow of a supersonic jet from a vessel with plane walls. Doklady Akad. Nauk SSSR (N.S.) 58, 381–384 (1947). (Russian)

In a previous paper [Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] 9, 121–143 (1945); translation in Tech. Memos. Nat. Adv. Comm. Aeronaut., no. 1155 (1947); these Rev. 7, 496; 8, 541] the author reduced the problem of the flow, with maximum possible flux, of a supersonic jet from a vessel with plane walls to a Tricomi boundary value problem for Chaplygin's equation for the stream function,

$$\frac{\partial}{\partial\tau}\left[\frac{2\tau}{(1-\tau)^{\beta}}\frac{\partial\psi}{\partial\tau}\right]+\frac{1-(2\beta+1)\tau}{2\tau(1-\tau)^{\beta+1}}\frac{\partial^2\psi}{\partial\theta^2}=0,$$

where $\tau=w^2/w_m^2$ (w is the speed, w_m the speed at zero pressure), $\beta=1/(\gamma-1)$, $\gamma=c_p/c_v$ and θ is the angle of the velocity vector. The present paper contains the results of numerical computations in the case of a vessel with plane walls, symmetrically placed with respect to the x-axis, and making angles of $\pm\pi/2$ with it. The value 0.85 was obtained for the flux coefficient $\rho_0 Q/(2b\rho^*a^*)$, where $2b$ is the width of the aperture, $\rho_0 Q$ is the given maximum flux, ρ_0 the density far inside the vessel, and ρ^* and a^* are the sonic density and speed, respectively. This is compared with the value 0.74 given by Chaplygin [Učenye Zapiski Imp. Moskov. Univ., Otd. Fiz.-Mat. 21, 1–121 (1904); translated in Tech. Memos. Nat. Adv. Comm. Aeronaut., no. 1063 (1944); these Rev. 7, 495] for the limiting case of a subsonic jet with sonic free streamlines. *J. B. Diaz* (Providence, R. I.).

Garrick, I. E., and Kaplan, Carl. On the flow of a compressible fluid by the hodograph method. I. Unification and extension of present-day results. Tech. Rep. Nat. Adv. Comm. Aeronaut., no. 789, 24 pp. (1944).

Garrick, I. E., and Kaplan, Carl. On the flow of a compressible fluid by the hodograph method. II. Fundamental set of particular flow solutions of the Chaplygin differential equation. Tech. Rep. Nat. Adv. Comm. Aeronaut., no. 790, 21 pp. (1944).

Particular solutions of the hodograph equations for plane steady isentropic air flow ($\gamma=1.4$) are given, together with a full discussion of "velocity correction formulae" suggested by the theory. Curves for practical use of the formula most favoured by the authors are given in a convenient form. The hypergeometric functions of Chaplygin [Učenye Zapiski Imp. Moskov. Univ., Otd. Fiz.-Mat. 21, 1–121 (1904); translated in Tech. Memos. Nat. Adv. Comm. Aeronaut., no. 1063 (1944); these Rev. 7, 495] are tabulated for certain positive and negative suffixes; and for certain negative suffixes, for which the Chaplygin function does not exist, an alternative solution is tabulated. With each function its logarithmic derivative is tabulated. Some of the properties of limit lines are obtained. *M. J. Lighthill.*

Prim, R. C. On a family of rotational gas flows. Quart. Appl. Math. 6, 319–325 (1948).

The author gives solutions for rotational two-dimensional steady flows of a perfect gas of the following type. Let v_r be the radial component of the velocity and v_θ the circumferential component and let a be the local value of the maximum or "ultimate" velocity magnitude attainable by expansion to zero pressure, a constant along any stream line. These solutions are

$$w_r=v_r/a=u(\theta)=D\sin m(\theta-\theta_0),$$
$$w_\theta=v_\theta/a=v(\theta)=C\cos m(\theta-\theta_0),$$

where C, D and m are constants to be determined from the equations of motion. The author thus obtains solutions similar to the well-known irrotational Prandtl-Meyer flow. However, in the author's rotational solutions, the radial lines are not the characteristic lines of the differential equations and therefore the useful "patching" property of the Prandtl-Meyer solution is absent. *H. S. Tsien.*

*Ehlers, F. E. **Methods of linearization in compressible flow. I. Janzen-Rayleigh method.** Prepared under the supervision of G. F. Carrier. Tech. Rep. no. **F-TR-1180A-ND (GDAM A-9-M IV/I).** Headquarters Air Materiel Command, Wright Field, Dayton, Ohio. iv+101 pp. (1948).

This is a faithful account of previous works on the Janzen-Rayleigh method of expanding the solution of uniform compressible flow over a body into a series in M^2 (M is the Mach number of the undisturbed stream). The report's chief usefulness is thus in the presentation of results of somewhat inaccessible papers. There is, however, one exception: the author has extended Imai's method of residues for the velocity distribution on the contour of an arbitrary cylinder to second order terms in M^4. *H. S. Tsien.*

*Roberts, R. C. **The method of characteristics in compressible flow. II. (Unsteady flow).** Prepared under the supervision of C. C. Lin. Tech. Rep. no. **F-TR-1173D-ND (GDAM A-9-M II/2).** Headquarters Air Materiel Command, Wright Field, Dayton, Ohio. vii+90 pp. (1947).

[For part I, by J. S. Isenberg, cf. the same Rep. no. **F-TR-1173A-ND (GDAM A-9-M II/1)** (1947); these Rev. 10, 74.] This work is concerned with the application of the method of characteristics to unsteady flows in one space variable. The theory thus covers unsteady flows in pipes of constant or slowly varying cross-section, also cylindrically and spherically symmetrical flows. The numerical finite difference procedures described apply to both isentropic and nonisentropic flows, the graphical methods only to the isentropic flows. The various methods differ as well in the dependent variables employed, in their accuracy, and in

their adaptability to different types of flow problems. The procedures are shown for various boundary conditions, e.g., discharge from a pipe into a reservoir at constant pressure, and for flows with shock waves. However, the procedure for strong unsteady shocks is not included. The appendix contains several illustrative examples, with diagrams, of the procedures and types of flows described in the text. The bibliography seems to be complete. *D. Gilbarg.*

de Haller, P. On a graphical method of gas dynamics. Ministry of Aircraft Production [London], R.T.P. Translation no. 2555, 24 pp. (1945).

The original appeared in Tech. Rundschau Sulzer 1945, no. 1, 6–24, and an English translation in Sulzer Tech. Rev. 1945, no. 1, 6–24; these Rev. 8, 611.

Kudriashov, L. K. Plane parallel gas flow past an ellipse. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] 11, 119–128 (1947). (Russian. English summary)

The author describes a method for the determination of a subsonic flow pattern of a compressible fluid past an oval-shaped domain. Following Nekrasov [same journal 8, 249–266 (1944); these Rev. 7, 38] he applies the Legendre transformation to obtain for the function

$$\Phi(\tau,\theta)\equiv\Phi(u,v)=-\varphi(x,y)+xu+yv$$

the equation $\Delta\Phi=l_\infty^2[k^2\tau^2\Delta\Phi-(k^2-1)\tau^4\partial^2\Phi/\partial\tau^2]$, where $\theta=\tan^{-1}(u/v)$ and $\tau=\tau(M)$ are Cartesian coordinates in the distorted logarithmic plane. Here M and θ are polar coordinates in the hodograph (u, v)-plane; $l_\infty=V_\infty/n$, $n^2=2(\gamma-1)^{-1}c_\infty^2(1+\frac{1}{2}(\gamma-1)M_\infty^2)$, $\Delta\Phi\equiv\tau^2\Phi_{\tau\tau}+\tau\Phi_\tau+\Phi_{\theta\theta}$, where V_∞ and c_∞ are the speed and the speed of sound at infinity. Introducing complex solutions of the form $H=\Phi+i\Psi$, he writes $H=\sum l^{2n}H^{(n)}$, where the $H^{(n)}$ have to be successively determined from an infinite sequence of Poisson's equations. Choosing H_0, the corresponding function of an incompressible flow around an ellipse, he obtains a compressible fluid flow around an oval-shaped domain.

The reviewer wishes to add that the method can be considered as a modified integral operator procedure. [In this connection, cf. the following review.] *S. Bergman.*

Bergman, Stefan, and Epstein, Bernard. Determination of a compressible fluid flow past an oval-shaped obstacle. J. Math. Physics 26, 195–222 (1948).

The authors propose the extensive use of computational devices to solve plane compressible-flow problems, especially when the hodograph transformation is employed. As a guide to the preparation of extensive tables for this work, the authors have computed tables of certain functions used in the "integral-operator" method of Bergman [cf., e.g., Tech. Notes Nat. Adv. Comm. Aeronaut., no. 1096 (1946); these Rev. 8, 541, and the following review], namely, the first six functions designated $Q^{(n)}$. In this method, solutions of the (linear) differential equation for the stream function in the hodograph plane are obtained by applying an integral operator to solutions of the corresponding incompressible-flow equation, which is simpler. As an illustrative example, the flow about a symmetrical oval obstacle is calculated, first in incompressible flow, then in compressible. These calculations are presented in the form of tabular data. In a closing section, questions of continuing the solutions outside the domain of convergence of the operator, and of continuing into supersonic-flow regions, are discussed briefly.

W. R. Sears (Ithaca, N. Y.).

Bergman, Stefan. Two-dimensional subsonic flows of a compressible fluid and their singularities. Trans. Amer. Math. Soc. 62, 452–498 (1947).

The first part of the paper contains an exposition of some previous results of the author obtained by the hodograph method. In the following a pseudo-logarithmic plane (θ, λ) is used in which the potential φ and the stream function ψ satisfy the equations (*) $\varphi_\theta=l^k\psi_\lambda$, $\varphi_\lambda=-l^k\psi_\theta$, where $l=l(\lambda)(1-M^2)\rho^{-2}$. The author discusses the fundamental solutions φ and ψ of (*) and the periods of the corresponding many-valued associated functions. These facts were implicitly contained in a generalization of Cauchy's integral formula, as given by the author in his lectures [The Hodograph Method in the Theory of Compressible Fluid, Brown University, 1942, pp. 16–18]. The author also gives necessary and sufficient conditions in order that a hodograph possessing the singularities under consideration leads to a flow in the physical plane around a closed contour. Numerous figures give an illustration of the correspondence between the different planes. *A. Weinstein.*

Lin, C. C., Reissner, E., and Tsien, H. S. On two-dimensional non-steady motion of a slender body in a compressible fluid. J. Math. Physics 27, 220–231 (1948).

Beginning with the potential equation for plane isentropic nonstationary motion, the authors assume steady undisturbed flow far upstream and consider the case of small perturbations; i.e., the departure of velocity components and pressure from free-stream conditions is small. This leads to a partial differential equation, boundary conditions, and a pressure formula, which involve four parameters in addition to the stream Mach number: δ, the thickness ratio of the airfoil; k, the dimensionless frequency; λ, which denotes the relative magnitude of the transverse (v) and streamwise (u) disturbance velocities, and ϵ, which gives the order of magnitude of u compared to the stream speed. This result is then specialized for various subcases, namely (1) transonic flow, (2) the usual subsonic or supersonic flow, (3) and (4) two cases of high-frequency oscillations, and (5) quasi-stationary incompressible flow. In only the first of these cases is the approximate differential equation, obtained by neglecting small-order terms, nonlinear. This result is discussed; for example, if an airfoil is accelerated rapidly through sonic speed, the breakdown of the linearization is avoided by the rapid time variation.

The consequences as regards similarity in wind-tunnel testing are pointed out. It is also shown that the hypersonic case is excluded from the framework of small-perturbation flows, and large pressure disturbances must be expected. This case is analyzed, as well as a third highly-oscillatory case, involving large pressure disturbances. The similarity parameters for these cases are found. In the appropriate cases the results agree with those of von Kármán [J. Math. Physics 26, 182–190 (1947); these Rev. 9, 217], Kaplan [Tech. Note Nat. Adv. Comm. Aeronaut., no. 1527 (1948); these Rev. 9, 477], and Tsien [J. Math. Physics 25, 247–251 (1946); these Rev. 8, 237]. *W. R. Sears* (Ithaca, N. Y.).

Maslen, Stephen H. Method for calculation of pressure distributions on thin conical bodies of arbitrary cross section in supersonic stream. Tech. Notes Nat. Adv. Comm. Aeronaut., no. 1659, 19 pp. (1948).

Taking the undisturbed flow direction as the axis of the spherical coordinate system, the author first obtains, by using the linearized equation, the potential of a line source

distributed along any radial line with the azimuth angle m and the meridian angle δ, and with the local strength proportional to the distance from the origin. By using several of these oblique line sources, the flow over a cone of arbitrary section, yawed or not, can be approximated by an appropriate choice of four parameters: m, δ, the strength of the sources and the number of sources. Several numerical examples are given for bodies of various shapes at zero angle of attack. *H. S. Tsien* (Cambridge, Mass.).

Lagerstrom, P. A. Linearized supersonic theory of conical wings. Tech. Notes Nat. Adv. Comm. Aeronaut., no. 1685, viii+161 pp. (1948).

This report, which was originally published as a progress report of the Jet Propulsion Laboratory of the California Institute of Technology, develops the idea of "conical" flows introduced by Busemann [Schr. Deutsch. Akad. Luftfahrtforschung **7B**, 105–121 (1943); also translated in Tech. Memos. Nat. Adv. Comm. Aeronaut., no. 1100 (1947); these Rev. **8**, 415]. The differential equation is that of linearized supersonic irrotational flow: (1) $g_{x_1x_1}+g_{x_2x_2}-g_{x_3x_3}=0$, where g represents the velocity potential or one of the Cartesian velocity components, x_1, x_2, x_3 are rectangular Cartesian coordinates, x_3 being taken in the direction of the undisturbed stream, and, as is generally the case in this report, the stream Mach number has been put equal to $\sqrt{2}$ for simplicity. A flow is conical with respect to a point P if its velocity components, pressure, etc., are constant along rays issuing from P. It is proved that such a flow results if the boundary conditions are conical. The point P is then taken as the origin of coordinates. A transformation attributed to Chaplygin, $R=\{1-(1-r^2)^{\frac{1}{2}}\}/a$, $\Theta=\theta$, where $\theta=\tan^{-1}(x_2/x_1)$ and $r^2=x_1^2+x_2^2$, transforms (1) into Laplace's equation in a plane. Following Busemann, complex coordinates are introduced and the velocity components are written as the real parts of analytic functions. The usual boundary conditions at the Mach cone emanating from the origin, i.e., continuity of velocity components, are assumed without discussion.

Flat symmetrical wings and flat lifting wings of no thickness are first treated, in various classes of planforms, all exhibiting the conical character. Considerable use is made of function-theoretical methods, such as continuation of functions by reflection in boundaries, determination of functions by their singularities, etc. There is some discussion of the uniqueness of solutions. The Lorentz (oblique) transformation is used to obtain the solutions for raked tips from those for square tips; this presents some difficulties when there is a subsonic trailing edge. Here the Kutta-Joukowski condition of finite velocities is imposed. The camber of constant-lift wings is also investigated, as are two cases where the lift is prescribed over part of the wing, the angle of incidence over the rest of it. Finally, attention is given to wings having piece-wise flat surfaces, in a conical sense, as are encountered in some control-surface problems.
W. R. Sears (Ithaca, N. Y.).

Robinson, A. On some problems of unsteady supersonic aerofoil theory. Coll. Aeronaut. Cranfield. Rep. no. 16, 21 pp. (1 plate) (1948).

In the first part of the paper, the author studies the two-dimensional problem of linear accelerated supersonic motion of a symmetrical airfoil at zero angle of attack. The velocity potential is built up from a source distribution. The pressure acting on the surface of the airfoil is shown to consist of three terms. The first term is the steady motion term, as obtained by Ackeret's theory. The second term depends on the square of the velocity during a limited period preceding the instant under consideration. The third term depends on the acceleration and may be considered to be the apparent mass effect. However, it is shown that when the square root of the product of the chord length of the airfoil and the acceleration is small in comparison to the velocity of sound, the sum of the second and the third terms is negligible compared with the first term. Thus for all accelerated supersonic motions (not transonic motion) which are likely to occur in practise, a quasi-steady calculation is sufficient.

In the second part, the author considers the oscillation of a delta-wing about the apex, with the leading edge of the wing behind the Mach cone. By separating variables, the normal solutions are obtained as products of Bessel functions and Lamé functions. It is shown that the velocity potentials corresponding to the vertical and the pitching oscillations can be represented by series of these normal solutions. No numerical results are given.
H. S. Tsien (Cambridge, Mass.).

Miles, John W. The aerodynamic forces on an oscillating flap at supersonic speeds. J. Aeronaut. Sci. **15**, 565–568 (1948).

Using the general results developed in an earlier paper [same J. **14**, 351–358 (1947); these Rev. **8**, 610] the complex flutter coefficients are found. *Extract from the paper.*

Jones, W. Prichard. Aerodynamic forces on wings in non-uniform motion. Ministry of Supply [London], Aeronaut. Res. Council, Rep. and Memoranda no. 2117 (8928), 36 pp. (1945).

Pistolesi, E. Forze e momenti in una corrente leggermente curva convergente. Pont. Acad. Sci. Comment. **8**, 609–631 (1944).

The author obtains the forces and couples on a body in a steady stream of incompressible inviscid fluid, which is non-uniform but only to such an extent that, in the region where the body's presence is principally felt, its potential may be approximated by quadratic and linear terms. His method differs slightly from Tollmien's [Ing.-Arch. **9**, 308–326 (1938)] and his expression for the couples is different: his explanation of Tollmien's error seems satisfactory.
M. J. Lighthill (Manchester).

Pistolesi, Enrico. Sull'interferenza di una galleria aerodinamica a contorno misto. II. Pont. Acad. Sci. Comment. **7**, 75–100 (1943).

The first part of this paper [same Comment. **4**, 321–341 (1940); these Rev. **2**, 330] gives a method of determining the interference of a circular wind-tunnel on an airfoil, where the trailing vortices are represented by a two-dimensional doublet. The author now applies the same method to the problem where the vortices are to be represented (i) by a pair of two-dimensional vortices, (ii) by a line of vortices "elliptically" distributed. The circular wall is composed of free and rigid portions, and the method consists in finding the complex velocity $w(z)$, which satisfies the boundary conditions that $zw(z)$ is real on the free portions and pure imaginary on the rigid portions. The problem is first solved for the case where the rigid portions consist of two equal arcs symmetrically placed round the circumference, and it is then generalised to deal with the case of any even number of equal rigid arcs symmetrically situated on the circular wall. *A. J. McConnell* (Dublin).

Ferrari, Carlo. Un metodo rapido approssimato di calcolo delle caratteristiche aerodinamiche delle schiere di profili alari. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **81–82**, 264–283 (1948).

The approximate method given, of calculating the potential flow at small incidence past a cascade of congruent aerofoils of small thickness and camber, is based on the theory of slightly nonuniform flow past isolated aerofoils.

M. J. Lighthill (Manchester).

Roy, Maurice. Écoulement théorique et médian dans un rotor à palettes radiales. C. R. Acad. Sci. Paris **227**, 496–498 (1948).

The flow of an inviscid incompressible fluid in a rotating rotor having N radial flat blades is discussed. The vanishing of the axial components of velocity at one cross-section of the rotor is taken as a boundary condition.

G. F. Carrier (Providence, R. I.).

Marble, Frank E. The flow of a perfect fluid through an axial turbomachine with prescribed blade loading. J. Aeronaut. Sci. **15**, 473–485 (1948).

Assuming a turbine with an infinite number of blades, the author idealizes the flow as an axially symmetric one which takes place between the two circular cylinders $r=r_1$ and $r=r_2$, $0<r_1<r_2$. He denotes by $u(r, z)$, $v(r, z)$ and $w(r, z)$ the radial, tangential and axial velocity components; by $\xi(r, z)$, $\eta(r, z)$ and $\zeta(r, z)$ the corresponding vorticity components; by F_r, F_θ and F_z the corresponding components of the force applied by the blades to the fluid; by ω the angular velocity of the blades, and by w_0 the mean of w with respect to r. To linearize the equations, he assumes that $|u|\ll|w|$, $|u(\partial\eta/\partial r)|\ll|w(\partial\eta/\partial z)|$ and $|w-w_0|\ll|w_0|$. From the continuity equation and the dynamic equations he then derives the approximate equation for $\bar{u}=u/w_0$,

$$\frac{\partial^2\bar{u}}{\partial r^2}+\frac{\partial}{\partial r}\left(\frac{\bar{u}}{r}\right)+\frac{\partial^2\bar{u}}{\partial z^2}=\frac{\partial}{\partial z}f(r, z),$$

where $f(r, z)=[(\bar{v}_0/r)-(\omega/w_0)][(\partial/\partial r)(\bar{v}_0 r)]+(F_r/w_0^2)$ and

$$w_0[\bar{v}_0(r, z)-\bar{v}_0(r, -\infty)]=\int_{-\infty}^{z}[F_\theta(r, \beta)/w_0]d\beta.$$

For $\bar{v}=v/w_0$ and $\bar{w}=w/w_0$, he likewise obtains the approximate equations

$$\bar{v}-\bar{v}_0=-\int_{-\infty}^{\infty}\bar{u}(\partial/r\partial r)(r\bar{v}_0)d\beta,\quad \bar{w}-1=\int_{-\infty}^{z}(\partial/r\partial r)(r\bar{u})d\beta.$$

As boundary conditions he takes $\bar{u}(r, \pm\infty)=\bar{u}(r_1, z)=\bar{u}(r_2, z)=0$, $\bar{v}(r, -\infty)=g(r)$ and $\bar{w}(r, -\infty)=1$. He then expresses the solutions for $\bar{u}$, $\bar{v}$ and $\bar{w}$ as definite integrals involving three Green's functions which are defined by infinite series involving linear combinations of the Bessel functions of orders zero and one. Finally, he illustrates his results by considering certain special blade loadings, corresponding to which he also gives some exponential approximations to $\bar{u}$, $\bar{v}$ and $\bar{w}$. *M. Marden* (Milwaukee, Wis.).

Bugaenko, G. A. On the theory of hydrodynamical grids with thin wings of arbitrary shape. Akad. Nauk SSSR. Prikl. Mat. Meh. **12**, 453–462 (1948). (Russian)

The author considers an incompressible potential two-dimensional flow past a straight cascade of infinitely thin slightly curved profiles. He assumes that the chords of the profiles are either situated along the axis of the cascade or else are perpendicular to the axis. The flow is represented as being due to a doublet at infinity and to vortices distributed along the chords of the profiles. For the density of the vortex-distribution the author derives a singular linear integral equation (i.e., an integral equation with Cauchy principal values). Using complex variable methods this equation can be solved in closed form. The resulting formulas for the pressure distribution contain as a limiting case Munk's formulas. The method is then extended to profiles of small but not vanishing thickness. In that case the flow is considered as being produced by a doublet at infinity and by vortices, sources, and sinks distributed along the chords. *L. Bers* (Syracuse, N. Y.).

Ray, M. On a type of non-stationary turbulent wake. Bull. Calcutta Math. Soc. **39**, 139–142 (1947).

The author considers a uniform stream u_0 disturbed by a symmetrical cylinder. If u_0-u is the mean velocity, parallel to the undisturbed flow, far downstream, certain assumptions concerning the dependence of u and the mixture length on time and place lead to

$$u/u_0\propto y_0^{\frac{1}{2}}\{1-(y/y_0)^{\frac{1}{2}}\}\exp[-2c(x+u_0t)],$$

where $2y_0$ is the breadth of the wake at abscissa x and c is a constant. A similar result is obtained in the case of a solid of revolution. The cases considered represent the dying out of a steady wake. They offer considerable difficulty of interpretation. *L. M. Milne-Thomson* (Greenwich).

Schuh, H. Einige Probleme der Strömung, des Wärmeüberganges und der Diffusion bei Laminarströmung längs einer ebenen Platte. Österreich. Ing.-Arch. **2**, 346–360 (1948).

A shorter version appeared in Z. Angew. Math. Mech. **25/27**, 54–60 (1947); these Rev. **9**, 393.

Fainzil'ber, A. M. A generalization of the theory of the "mixing line" in the flow around curved profiles. Doklady Akad. Nauk SSSR (N.S.) **58**, 555–558 (1947). (Russian)

The author extends the Prandtl-Nikuradze law for the turbulent boundary layer velocity distribution to the case where the pressure gradient is not zero. The velocity distribution is obtained in closed form and contains the Prandtl-Nikuradze law as a special case. This result is then used to obtain a simple expression for the drag coefficient and both results are shown to give good numerical agreement with several experimental investigations.

P. Chiarulli (Providence, R. I.).

Konakov, P. K. The coefficient of resistance for smooth tubes. Izvestiya Akad. Nauk SSSR. Otd. Tehn. Nauk **1948**, 1029–1036 (1948). (Russian)

L'auteur étudie le mouvement turbulent isotherme plan stationnaire d'un fluide visqueux incompressible dans un tuyau circulaire. En négligeant les termes de la viscosité à l'intérieur du noyau turbulent du courant et en introduisant quelques hypothèses simplificatrices, il trouve une relation simple entre le coefficient de résistance et le logarithme du nombre de Reynolds. La formule est en assez bon accord avec les résultats de mesure de Nikuradze. En terminant, l'auteur réfute une critique de Kolmogoroff.

M. Kiveliovitch (Paris).

Nevzglyadov, V. G. The theory of the turbulent motion of compressible fluids. Doklady Akad. Nauk SSSR (N.S.) 58, 547–550 (1947). (Russian)

In this paper the averaging procedure of Reynolds is applied to the general Navier-Stokes equations for a compressible fluid including thermal heat conduction. As in the classic incompressible case correction terms to the equations for the mean behavior of the fluid are found. In formulating a functional behavior for these correction terms the author considers only the turbulent energy and not the scale of turbulence (i.e., the "mixing length"), and the results are expressed in terms of dimensional constants to be obtained by experiment. In the opinion of the reviewer this functional formulation of the corrective terms is invalid.

W. D. Hayes (Providence, R. I.).

Ursell, F. On the waves due to the rolling of a ship. Quart. J. Mech. Appl. Math. 1, 246–252 (1948).

In a previous paper [Proc. Cambridge Philos. Soc. 43, 374–382 (1947); these Rev. 9, 117] the author has discussed the effect of a fixed vertical barrier on surface waves in deep water. He now uses the same methods to determine the waves generated in water of infinite depth by a vertical plate which describes a forced rigid periodic motion about a horizontal axis. The motion of the plate and the liquid is assumed to be small enough to permit one to use the linearized form of the boundary conditions. The plate is to extend to a finite depth below the surface of the liquid and be wide enough to make the resulting motion of the liquid two-dimensional. Explicit expressions for the motion of the liquid are obtained by solving a pair of integral equations for the horizontal velocity in the vertical plane containing the plate. The amplitude of the resulting waves is expressed in terms of Bessel functions. The results are applied to waves generated by the rolling of a ship and to the waves generated by wave-makers in experimental tanks.

F. John (New York, N. Y.).

Groen, P. Two fundamental theorems on gravity waves in inhomogeneous incompressible fluids. Physica 14, 294–300 (1948).

The author proves two theorems concerning simple harmonic waves of small amplitude in incompressible fluid whose density at depth z is $\rho_0(z)$ in the undisturbed state. (I) The periods of the waves are monotonic increasing functions of the wave length. (II) There exists a lower bound of the periods which is 2π times the minimum value of $\sqrt{(\rho_0/g\rho_0')}$, this being the limit of the periods when the wave length tends to zero. *L. M. Milne-Thomson.*

Groen, P. Contribution to the theory of internal waves. Koninklijk Nederlands Meteorologisch Instituut de Bilt (Nederland), no. 125. Mededelingen en Verhandelingen, Ser. B, part II, no. 11, 23 pp. (1948).

The author considers incompressible inviscid fluid of density continuously variable with depth and extending from $z=-\infty$ to $z=\infty$ vertically. The fluid is at rest in the unperturbed state and the internal waves are treated as small perturbations. When the law of variation of the specific volume $S(z)$ is taken as $S(z)=S_0+\frac{1}{2}(\Delta S)\tanh(2z/b)$, where ΔS is the total variation of specific volume and b is regarded as a measure of the thickness of the transition layer, it is found that the wave-length $2\pi\lambda$ and the period $2\pi\tau$ are related by

$$(g\Delta S/bS_0)\tau^2=n(n+1)(2\lambda/b)^2+(2n+1)2\lambda/b+1.$$

Here $n=0, 1, 2, \cdots$ and represents the order of the mode of oscillation. Thus when $\lambda\to 0$ the period approaches a minimum value. The existence of this minimum value appears to be a general feature not restricted to the above law of density. In the case of a rotating earth the author finds the same relation as that given above to exist between τ and $\lambda(1-4\omega^2\tau^2)^{\frac{1}{2}}$, where ω is the vertical component of the angular velocity. *L. M. Milne-Thomson* (Greenwich).

Dean, W. R. On the reflexion of surface waves by a submerged circular cylinder. Proc. Cambridge Philos. Soc. 44, 483–491 (1948).

The author has already considered the reflexion of surface waves by a submerged plane barrier [same Proc. 41, 231–238 (1945); these Rev. 8, 110]. In this paper the effect of a submerged circular cylinder is investigated. The complex potential is determined from the boundary conditions by mapping on a circular annulus and an approximation to a definite numerical example is worked out. It is found that at a distance from the obstacle there is no reflected wave, while the amplitude of the incident wave is unaltered but its phase is changed. This result is shown to hold in general.

L. M. Milne-Thomson (Greenwich).

Rombakis, Sokrates. Über die Phasengeschwindigkeit von Wellen endlicher Amplitude. Z. Meteorologie 1, 302–303 (1947).

The author claims to have derived an exact expression for the propagation velocity of surface waves of finite amplitude for infinite water-depth. The zeroth order and the first order terms are in agreement with Stokes's result. However, this agreement can only be accidental, as the method of deriving the expression for the propagation velocity by taking surface quantities at the undisturbed water level is wrong for any calculation on finite amplitude waves.

H. S. Tsien (Cambridge, Mass.).

*Geiringer, Hilda. **On numerical methods in wave interaction problems.** Advances in Applied Mechanics, edited by Richard von Mises and Theodore von Kármán, pp. 201–248. Academic Press, Inc., New York, N. Y., 1948. $6.80.

This paper presents an analysis, by numerical methods, of the interaction of a shock wave and a rarefaction wave in one-dimensional non-steady fluid flow. The problem is first considered under the assumption that one pressure-density relation is valid throughout: namely, $p/\rho^k=$constant. The solution up to the intersection of the shock and the first rarefaction line of the centered rarefaction wave can be handled analytically, of course, and is given in some detail. The fundamentally difficult problem of determining the shock wave in the interaction region, however, yields only to numerical analysis, and a description of the procedure employed is presented. The requirements are subsequently relaxed and the numerical methods extended to handle the situation in which an entropy increase across the shock is admitted.

A description is given of von Neumann's mechanical model of hydrodynamical motion which exhibits the analogy between nonlinear wave motion and movement of a finite number of particles joined by springs. The numerical integration of the partial differential equations of motion is carried out without regard to the possibility of a shock, and the shock is located afterward by the appearance of "superposed" oscillations of the particles.

The paper concludes with an indication of the procedure, proposed by von Mises, for introducing the effect of viscosity and hence for finding a continuous solution to the interaction problem. The possibility of extending these methods to more than one spatial dimension is also mentioned. *E. N. Nilson* (Hartford, Conn.).

Stanyukovič, K. P. The two-sided flow of a gas from a cylindrical vessel into a tube. Doklady Akad. Nauk SSSR (N.S.) **58**, 201–204 (1947). (Russian)

Consider a cylindrical vessel at whose ends are attached tubes. Let the gas in the cylinder be initially at a high pressure compared to that in the tubes and let the ends of the cylinder be opened at different times. The author discusses the waves which arise, and their interpretation. *G. F. Carrier* (Providence, R. I.).

Schaefer, M. Formation of envelopes of Mach waves on a tangentially struck blade. The Graduate Division of Applied Mathematics, Brown University. Translation **A9-T-11**, ii+29 pp. (1948).

[Translation of Technische Hochschule Dresden, Archiv **44/10** (1943).] The object is to formulate relations for the formation of an envelope by the Mach waves influenced by a concave boundary. It is shown that the conditions are the same as for the appearance of a "limiting line" in the hodograph method. The method adopted here is to assume an envelope originating at a known point (origin of coordinates) caused by the family of left-running Mach waves, to assume a right-running wave of a selected shape originating also at the origin, and to calculate the position and initial shape of the boundary. This is done, essentially, by expanding in powers of α the hodograph equations of isentropic irrotational plane flow, where α is the inclination of a Mach wave relative to the x-axis. The result shows that the stronger the curvature of the boundary, the nearer to it the envelope begins. This relation is independent of the assumed expression for the right-running Mach wave. The envelope, of course, does not exist in nature. The author assumes it is replaced by a shock wave beginning with infinitesimal strength at the same point, and proceeds to determine its initial curvature. This calculation gives the result that the shock wave curvature is $\frac{3}{4}$ of the initial curvature of the envelope, so that the shock wave actually lies behind the fictitious envelope. *W. R. Sears* (Ithaca, N. Y.).

Pack, D. C. The condition for the detachment of the shock wave from a wedge in a supersonic stream. Proc. Cambridge Philos. Soc. **44**, 298–300 (1948).

When a wedge is disposed symmetrically in a supersonic stream a shock wave is formed which is itself wedge-shaped and attached to the tip for sufficiently high Mach numbers but which for lower Mach numbers is curved and detached from the tip. Taylor and Maccoll in 1931 determined by graphical treatment of the shock equations the Mach number below which, for a given wedge angle, an attached shock configuration is impossible. The present author pursues the problem analytically and exhibits the critical shock wave angle as a root of a quartic equation. *D. P. Ling.*

Gutin, L. On the sound field of a rotating propeller. Tech. Memos. Nat. Adv. Comm. Aeronaut., no. **1195**, 21 pp. (1948).

Translated from Phys. Z. Sowjetunion **9**, 57–71 (1936).

Fox, E. N. The diffraction of sound pulses by an infinitely long strip. Philos. Trans. Roy. Soc. London. Ser. A. **241**, 71–103 (1948).

The author obtains the solution for the two-dimensional diffraction problem of a perfectly reflecting strip subject to a plane pulse of constant unit pressure at normal incidence. The problem is formulated as an inhomogeneous integral equation

$$(*)\quad \pi=\lambda\int_0^\infty A(y')K_0[\lambda|y-y'|]dy' +\lambda\int_0^\infty A(y')K_0[\lambda(1+y+y')]dy',\qquad y>0,$$

where $A(y')$ is the function desired and is related to the pressure function, and K_0 is Macdonald's Bessel function of order zero. Upon noting that the absence of the second term in the above integral equation provides an integral equation of the Wiener-Hopf type, a method of successive substitutions is suggested as a means of solution for (*). Excitation pulses of different shape are also discussed and some of the results are presented in numerical form. The above methods have been applied to several other two-dimensional diffraction problems and these results are promised in a later paper. *A. E. Heins* (Pittsburgh, Pa.).

Spence, R. D. The diffraction of sound by circular disks and apertures. J. Acoust. Soc. Amer. **20**, 380–386 (1948).

The author studies the diffraction of sound by circular disks and apertures. Confining himself to the case of a plane monochromatic incident wave traveling in the axial direction, he expands the scattered velocity potential into a series of products of zero-order oblate-spheroidal wave functions, viz., $\psi=\sum A_l u_l(\eta)\,{}^{(3)}v_l(\xi)$. The angular wave functions $u_l(\eta)$ are presented in the form of power series with coefficients satisfying a homogeneous linear difference equation of the second order, e.g., for l even, $u_l(\eta)=\sum_{r=0}^{\infty}c_{2r}(1-\eta^2)^r$. The radial functions (representing divergent waves) are defined as a combination of two standing-wave radial functions, ${}^{(3)}v_l(\xi)={}^{(1)}v_l(\xi)+i\,{}^{(2)}v_l(\xi)$, in which ${}^{(1)}v_l(\xi)=\sum_{r=0}^{\infty}a_{2r}\xi^{2r}$,

$${}^{(2)}v_l(\xi)=Q_l\left[(\tan^{-1}\xi-\tfrac{1}{2}\pi)\,{}^{(1)}v_l(\xi)+\sum_{r=0}^{\infty}b_{2r+1}\xi^{2r+1}\right],$$

where the a's satisfy a homogeneous linear difference equation of the second order, while the b's follow from a similar inhomogeneous equation depending on the a's; Q_l is a constant of normalization. The radial functions are related to the angular functions by

$${}^{(1)}v_l(\xi)=\tfrac{1}{2}q_l\int_{-1}^{1}e^{i\epsilon\xi\eta}u_l(\eta)d\eta,$$

$${}^{(3)}v_l(\xi)=q_l\int_{ic}^{1}e^{i\epsilon\xi\eta}u_l(\eta)d\eta,$$

where $\epsilon/2\pi$ is the radius of the obstacle divided by the wavelength and q_l is a constant of proportionality. Expansions in terms of spherical Bessel and Hankel functions follow at once from these integral representations. Expressions are given for the scattering cross-sections of the disk and the aperture; numerical results are plotted in diagrams for $0\leqq\epsilon\leqq5$. During the preparation of the paper the author was unaware of similar work by the reviewer [Thesis, Groningen, 1941; these Rev. **8**, 179] including numerical results for $0\leqq\epsilon\leqq10$. *C. J. Bouwkamp* (Eindhoven).

Miles, John W. The diffraction of a sound wave by an infinite set of plates. J. Acoust. Soc. Amer. 20, 370–374 (1948).

It was suggested by Schwinger that the special class of electromagnetic boundary value problems where a plane wave is incident upon a number of semi-infinite perfectly conducting parallel planes or cylinders can all be formulated as Wiener-Hopf integral equations and solved exactly. The case of an infinite set of parallel staggered semi-infinite plates was discussed in detail by Carlson and Heins [Quart. Appl. Math. 4, 313–329 (1947); 5, 82–88 (1947); these Rev. 8, 422, 614]. The problem of the diffraction of sound waves by a similar structure is equivalent to the electromagnetic problem discussed in the second Carlson-Heins paper. It is discussed in the present paper, and the conditions for plane wave propagation within the plates, for a single reflected wave, and expressions for the reflection and transmission coefficients are given. *E. T. Copson* (Dundee).

Elasticity, Plasticity

Diaz, J. B., and Greenberg, H. J. Upper and lower bounds for the solution of the first boundary value problem of elasticity. Quart. Appl. Math. 6, 326–331 (1948).

The authors deal with the first boundary value problem of three-dimensional elasticity, i.e., with Navier's equations $\mu w_{i,jj}+(\lambda+\mu)w_{j,ji}+X_i=0$ in R $(i, j=1, 2, 3)$, w_i given on S, where R is a domain bounded by a surface S, w_i is the ith component of the displacement, λ and μ are Lamé's constants, X_i is the ith component of body force, commas indicate partial differentiation, and repeated subscripts indicate summation over the range 1, 2, 3. Introducing the notations

$$(\varphi, \psi)=\int_R [\mu\varphi_{i,j}\psi_{i,j}+(\lambda+\mu)\varphi_{i,i}\psi_{j,j}]dR,$$

$$(\varphi, \psi)_R=\int_R \varphi_i[\mu\psi_{i,jj}+(\lambda+\mu)\psi_{j,ji}]dR,$$

$$(\varphi, \psi)_S=\int_S \varphi_i[\mu\psi_{i,j}n_j+(\lambda+\mu)\psi_{j,j}n_i]dS,$$

where n_j is the jth component of the outer normal to S, the authors seek to obtain for a given point ξ_1, ξ_2, ξ_3 upper and lower bounds for the numbers $w_i(\xi_1, \xi_2, \xi_3)$, $i=1, 2, 3$. They obtain $(w_k(\xi_1, \xi_2, \xi_3)-b_k)^2\leqq ac_k$, $(w_k(\xi_1, \xi_2, \xi_3)-b_k')^2\leqq ac_k$, where

$$a=(u-v, u-v),$$
$$b_k=-(u, U^{(k)})-(U^{(k)}, u)-(Z^{(k)}, w)-(w, Z^{(k)})_S,$$
$$c_k=(U^{(k)}-V, U^{(k)}-V),$$
$$b_k'=(v, V^{(k)})-(W^{(k)}, v)-(w, V)_S-(Z^{(k)}, w)-(w, Z^{(k)})_S,$$

and

$$\mu v_{i,jj}+(\lambda+\mu)v_{j,ji}+X_i=0$$

in R, $u_i=w_i$ on S;

$$\mu V_{i,jj}+(\lambda+\mu)V_{j,ji}=0$$

in R, $U_i^{(k)}=W_i^{(k)}=-Z_i^{(k)}$ on S;

$$\mu W_{i,jj}^{(k)}+(\lambda+\mu)W_{j,ji}^{(k)}=0$$

in R, $W_i^{(k)}=-Z_i^{(k)}$ on S. Finally, $W^{(k)}$, $U^{(k)}$, V are abbreviations for $(W_1^{(k)}, W_2^{(k)}, W_3^{(k)})$, $(U_1^{(k)}, U_2^{(k)}, U_3^{(k)})$, (V_1, V_2, V_3) and $Z_i^{(k)}$ are the "singular" solutions

$$Z_i^{(k)}=-\frac{\lambda+\mu}{8\pi(\lambda+2\mu)}\left[r_{,ki}-\frac{\lambda+2\mu}{\lambda+\mu}r_{,i}\delta_{ki}\right]$$

of Navier's equations with body forces zero which were given by Kelvin; r is the distance from (ξ_1, ξ_2, ξ_3) to (x_1, x_2, x_3) and δ_{ki} is the Kronecker delta. It is possible from these results to deduce two known minimum principles ("principle of minimum potential energy" and "Castigliano's principle"). *M. Pinl* (Cologne).

Elliott, H. A. Three-dimensional stress distributions in hexagonal aeolotropic crystals. Proc. Cambridge Philos. Soc. 44, 522–533 (1948).

Problems in three dimensions for hexagonal aeolotropic elastic materials (substances having five elastic constants) are reduced to the determination of two functions ϕ_1 and ϕ_2 of the coordinates x, y, z, which satisfy the equations $(\nabla_1^2+\partial^2/\partial z_i^2)\phi_i=0$ $(i=1, 2)$, where $\nabla_1^2=\partial^2/\partial x^2+\partial^2/\partial y^2$ and $z_i=z\nu_i^{-\frac{1}{2}}$, and ν_1 and ν_2 are the roots of a quadratic equation with coefficients depending on the elastic constants. The z-axis is considered to be normal to the hexagonal planes. The author also evaluates the displacements and stresses in terms of these functions and the elastic moduli. In case the problem possesses axial symmetry about the z-axis, the solution can be expressed in terms of a single function Φ, where $\partial\Phi/\partial z=\phi_1+\phi_2$. The governing equation reduces to $\nabla^4\Phi=0$ in the isotropic case. It should be noted that the constants ν_i may be complex and the functions ϕ_i conjugate complex. The cases of a single force along the z-axis at the origin and a doublet at the origin are worked out in detail. Graphs are given, in the first case, comparing the results obtained for zinc and magnesium as treated by this theory with the more usual computations based on isotropic elasticity. *G. H. Handelman* (Pittsburgh, Pa.).

Ferrandon, Jean. Résistance opposée par un continu élastique aux petits déplacements d'un solide. C. R. Acad. Sci. Paris 226, 2047–2048 (1948).

A rigid inclusion in an indefinitely large isotropic elastic medium is given a small displacement. The author concludes that the force exerted by the elastic medium on the inclusion is zero. His conclusion is based on the consequences of the incorrect assumption that the displacement may be taken as the gradient of a potential.
G. F. Carrier (Providence, R. I.).

Richter, H. Das isotrope Elastizitätsgesetz. Z. Angew. Math. Mech. 28, 205–209 (1948). (German. Russian summary)

This paper considers the relationship between stress and deformation in isotropic elastic materials subjected to finite strains, taking into account thermal expansion. Matrix notation is used, the deformation matrix $\mathfrak{A}$ being defined by $d\mathfrak{r}'=\mathfrak{A}d\mathfrak{r}$, $\mathfrak{r}$ being the initial, and $\mathfrak{r}'$ the displaced, position vector. Analysis of deformation into pure strain and rotation is effected and the most general relationship between stress and strain is deduced in terms of the stress and deformation matrices and the invariants of the deformation matrix. Similar expressions are evolved in terms of the logarithmic strain matrix, and the separated components representing change of shape and cubical dilatation. Restrictions are placed on these general expressions by assuming the existence of thermodynamic potentials. The possibility of applying Hooke's law for finite strains is considered,

defining strain as the pure deformation matrix minus the unit matrix, the so-called nominal strain. The conditions imposed by the existence of a thermodynamic potential are violated unless the Poisson's number $m=3$. A formulation in terms of the logarithmic strain matrix is not limited in this way. *E. H. Lee* (Providence, R. I.).

Rivlin, R. S. Large elastic deformations of isotropic materials. I. Fundamental concepts. Philos. Trans. Roy. Soc. London. Ser. A. 240, 459–490 (1948).

The author uses (x, y, z) notation, but for brevity in this review indicial notation will be used, the coordinates being rectangular Cartesians. This involves a modifying factor $\frac{1}{2}$ in the definition of shearing strains. In terms of displacement u_i, the author defines finite strain in the usual way: $e_{ij}=\frac{1}{2}(u_{i,j}+u_{j,i}+u_{k,i}u_{k,j})$, the comma denoting partial differentiation; he also employs the pure homogeneous portion of the strain: $e'_{ij}=\frac{1}{2}(u_{i,j}+u_{j,i}+u_{i,k}u_{j,k})$. [In matrix notation, $e=\frac{1}{2}(\bar{M}M-1)$, $e'=\frac{1}{2}(M\bar{M}-1)$, where $M_{ij}=\delta_{ij}+u_{i,j}$.] The paper is for the most part concerned with incompressible materials; such a material is called neo-Hookean if stress t_{ij} is related to e'_{ij} by $t_{ij}=(p+\frac{1}{2}E)\delta_{ij}+\frac{3}{2}Ee'_{ij}$, where p is a hydrostatic pressure and E a positive constant characterising the material; t_{ij} and p are measured at the displaced position which corresponds to the undisplaced position at which e'_{ij} are evaluated. The incompressibility relation reads (*) $|\delta_{ij}+2e'_{ij}|=1$ and so, by the stress-strain relations, the pressure is related to the stress by the cubic determinantal equation $|t_{ij}-p\delta_{ij}|=\frac{1}{27}E^3$. The roots of this cubic are examined and it is found that there is always one root, and only one, which is real and corresponds to a real displacement. The stored energy per unit volume is found to be $W=\frac{1}{2}Ee_{kk}$, and this is shown to be consistent with the usual formula when the strain is small. This formula agrees with that obtained by Treloar from statistical mechanics and is regarded as consistent with experimental results for vulcanized rubbers. Equations of motion are discussed by a variational principle. The equations of equilibrium read $-p_{,i}=\frac{1}{2}E(\Delta u_i+u_{j,i}\Delta u_j)$, where Δ is the Laplace differential operator. These three equations, together with the incompressibility relation (*), constitute four equations for p and u_i.

J. L. Synge (Dublin).

Rivlin, R. S. Large elastic deformations of isotropic materials. II. Some uniqueness theorems for pure, homogeneous deformation. Philos. Trans. Roy. Soc. London. Ser. A. 240, 491–508 (1948).

[Cf. the preceding review.] The problem here considered is that of the equilibrium of a cube of neo-Hookean (incompressible) material under the action of uniform normal stresses applied to its six faces. If the edges of the cube are originally of unit length and strain into lengths λ_1, λ_2, λ_3, and if the components of stress are assumed constant throughout the cube, the stress-strain relations and the incompressibility relation yield the following four equations: $\lambda_1 f_1=\frac{1}{2}E\lambda_1^2+p$, $\lambda_2 f_2=\frac{1}{2}E\lambda_2^2+p$, $\lambda_3 f_3=\frac{1}{2}E\lambda_3^2+p$, $\lambda_1\lambda_2\lambda_3=1$, where f_1, f_2, f_3 are the total loads on the faces. Here we have four equations for λ_1, λ_2, λ_3, p. It is shown that there is a unique solution corresponding to real deformation provided at least one of f_1, f_2, f_3 is negative (a compressional force), or if they are all positive and satisfy $f_1f_2f_3<\frac{1}{27}E^3$. The unique state is shown to be stable. If f_1, f_2, f_3 are all positive and satisfy $f_1f_2f_3>\frac{1}{27}E^3$, then there exist multiple solutions; these are analyzed into eight different types, some stable and some unstable. In the particular case $f_1=f_2=f_3=f$, there is a unique stable solution if $f<\frac{1}{2}E$. If $\frac{1}{2}E<f<2^{-1}E$, there are three states of stable equilibrium, none of which is identical with the undeformed state. If $2^{-1}E<f<\frac{3}{2}E$, there are four states of stable equilibrium, one of which is identical with the undeformed state. If $f>\frac{3}{2}E$, there is only one state of stable equilibrium, and this is identical with the undeformed state.

J. L. Synge (Dublin).

Rivlin, R. S. Large elastic deformations of isotropic materials. III. Some simple problems in cylindrical polar coordinates. Philos. Trans. Roy. Soc. London. Ser. A. 240, 509–525 (1948).

The first half of this paper consists of a translation into cylindrical coordinates of the basic formulae of part I [see the second preceding review]. The second half deals with torsion problems. For a right circular solid cylinder of radius a, the displacement under torsion is assumed to be a rigid rotation of each section about the axis of the cylinder through an angle ϕ proportional to the distance from one end ($\phi=\psi z$, $\psi=$constant). The sides of the cylinder being free from stress, the hydrostatic pressure is found to be $p=-\frac{1}{8}E\psi^2(a^2-r^2)-\frac{1}{2}E$. The torsional couple is $\frac{1}{2}\pi E\psi a^4$, and on each end there is a compressive force $\frac{1}{12}\pi E\psi^2a^4$. The theory is also applied to simultaneous torsion and extension of a solid right circular cylinder and to torsion of a hollow cylindrical tube. *J. L. Synge* (Dublin).

Sonntag, Gerh. Die Momentbelastung des Halbraumes. Z. Angew. Math. Mech. 28, 263–270 (1948). (German. Russian summary)

The known stress distributions for a point force acting at the plane surface of a semi-infinite body are used to determine stress distributions for torque nuclei, by taking the limit of the superposition of the stress distributions for equal and opposite forces with infinitesimally displaced lines of action. Straightforward superposition is not used, since in carrying out the process a stress distribution in cylindrical coordinates is differentiated partially with respect to one of the Cartesian coordinates, and the resulting change in the stress directions is not fully taken into account. The equilibrium equations are used to furnish the missing components. This would not have been necessary had Cartesian coordinates been used throughout, enabling a strict analysis of the superposition to be made. *E. H. Lee.*

Davis, E. A. A generalized deformation law. J. Appl. Mech. 15, 237–240 (1948).

The author proposes to establish a stress-strain law for isotropic materials which can be readily used to describe experimental data. To this end he restricts the discussion to conditions of loading under which the principal axes of strain remain fixed in the material and the ratios of the principal shearing strains are kept constant. Under these circumstances the use of a stress-strain law of the deformation type is legitimate, and the author suggests that such a law is most conveniently expressed by giving the functional relationships between (1) the octahedral shearing stress τ and the octahedral shear strain γ and (2) Lode's parameters μ and ν. [The most general deformation-type law for an isotropic material can indeed be specified by giving, say, γ and ν as functions of τ and μ. To assume, as the author does, that γ depends only on τ and ν only on μ means restricting the discussion to a rather special class of materials.]

W. Prager (Providence, R. I.).

Norzi, Livio. Teoria dell'instabilità elastica. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **81–82**, 253–263 (1948).

The author attempts to formulate conditions for elastic stability which are valid for all types of theories of elasticity. He suggests that instability occurs when the increase in internal elastostatic energy corresponding to a virtual displacement is exceeded by the virtual work F of the mass forces, so that a portion of the latter is left over to be converted into kinetic energy. The elastostatic energy may be expressed as the sum $L+S$, where L is the virtual work corresponding to a pure strain from an initially unstressed state and S is the virtual work of the internal forces. If the factor of stability ϕ relative to the given deformation be defined by $L+\phi(S-F)=0$, then the criterion of stability $L+S-F>0$ given above leads to the variational problem $|\phi|=$minimum as a necessary condition. Applying these notions to the classical situation, the author puts $2F=\int s_i U_{,i} s_j dV$, where s_i is the displacement vector and U is the potential of the mass forces; $2L=\int[\lambda(\epsilon_{ii})^2+2\mu\epsilon_{ij}\epsilon_{ij}]dV$, where the ϵ_{ij} are the strains; $2S=\int(\sigma_{ij}\omega_i\omega_j-\sigma_{ij}\omega_i\omega_j)dV$, where the σ_{ij} are the stresses and the ω_i are the infinitesimal rotations ($2\omega_i=\epsilon_{ijk}s_{j,k}$). Thus he obtains equilibrium equations which differ from the usual ones by the addition of a term $(-\frac{1}{2})\epsilon_{ijk}(\sigma_{il}\omega_j-\sigma_{ij}\omega_l)_{,k}$, thus generalizing a result of Southwell [Philos. Trans. Roy. Soc. London. Ser. A. **213**, 187–244 (1913)]. *C. Truesdell* (Washington, D. C.).

Weber, C. Zur nichtlinearen Elastizitätstheorie. Z. Angew. Math. Mech. **28**, 189–190 (1948).

The author establishes five assumptions which must be made in the non-linear theory of elasticity in order to obtain the stress-strain law which A. Philippidis proposed recently [Z. Angew. Math. Mech. **25/27**, 31–32 (1947); these Rev. **10**, 80]. *W. Prager* (Providence, R. I.).

Levi, F. Sugli stati di coazione elastica di origine viscosa. I. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **4**, 65–70 (1948).

Levi, F. Sugli stati di coazione elastica di origine viscosa. II. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **4**, 197–201 (1948).

The first paper is concerned with the influence of creep on the stresses in statically indeterminate concrete structures. Making a simple [but in the reviewer's opinion unrealistic] assumption concerning the creep under constant stress, the author shows in what manner the principle of least work must be modified when elastic deformation is accompanied by creep. In the second paper, shrinkage is treated along similar lines. *W. Prager.*

***Alfrey, Turner, Jr. Mechanical Behavior of High Polymers.** Interscience Publishers, Inc., New York, 1948. xiv+581 pp. $9.50.

This book fills a gap long obvious to all workers in the polymer field. From a utilitarian viewpoint, substances of high molecular weight are of interest largely because of their mechanical properties; yet, in spite of considerable and still expanding effort, scientific knowledge of mechanical behavior lags far behind advances in the chemistry of polymerization. This situation is of course due in large measure to the intrinsic difficulty of the subject, for polymers rarely behave as ideal elastic solids or Newtonian liquids, usually suffering (even under relatively mild stresses) large deformations which vary with time in a complicated way, so that even an inclusive phenomenological approach, to say nothing of molecular theory, still awaits development. However, the existing literature is widely dispersed, and the number of summary articles or books dealing with fundamentals is disappointingly small. The present volume will go far toward remedying this deficiency.

Alfrey's dual purpose, stated in his preface, is "to uncover the fundamental principles underlying the mechanical behavior of high polymers, and to show how such behavior is correlated with the molecular structures involved." No attempt is made to catalogue the properties of all known polymers, attention being confined to those (both natural and synthetic) which have received adequate scientific, as distinct from practical, investigation. The author, a physical chemist, has given a thorough mathematical treatment wherever feasible, but has not hesitated to sacrifice mathematical elegance for physical clarity or for the purpose of reaching a wider circle of readers. Thus, for example, tensor notation is not used in general, though it is briefly treated in an appendix. To give historical perspective, Alfrey has quoted the original literature far more extensively than is customary. This device is often successful, but occasionally an overlong and too detailed quotation detracts from the unity and fluidity of his presentation.

The work is divided into six chapters, each of which comprises a phenomenological and a molecular-structural discussion. The introductory chapter sets forth the existing general theories of elasticity and flow, and shows clearly the need for thorough treatments of large strains and nonlinear behavior. The second chapter, on plastoelastic behavior of amorphous linear high polymers, is the most fully developed in the book. Here, at least for small deformations, the phenomenological framework is complete, the author's own treatment of nonhomogeneous and combined stresses supplying the final steps. The outstanding problem of molecular theory here is the explanation of the inevitable broad spectrum of relaxation or retardation times displayed by such materials. Alfrey's qualitative discussion of this question is wholly admirable, but his attempt at quantitative expression is of questionable value.

In the next three chapters, the effects of additional structural features are considered in turn, these being crosslinking of chains into three-dimensional networks, crystallization and plasticization. Noteworthy are extensive sections on the statistical theory of rubberlike elasticity and the mechanism of network formation, and the description of the author's comprehensive studies of certain plasticized polymers. The discussion of ultimate strength and related properties, in the final chapter, necessarily abandons a quantitative approach, but the correlation of observed effects with molecular structure is highly illuminating.

In assessing the degree to which the author has fulfilled his aims, it is necessary to reemphasize the complexity of the subject and its relatively undeveloped state. Many of the details of this book will doubtless require extensive amplification or revision in a few years. Alfrey never hesitates, however, to point out frankly the shortcomings of present treatments, and to indicate at each stage the fundamental problems that remain to be solved. In summary, the book may be described as an up-to-date critical account of present fundamental knowledge of polymer mechanics, by one who has contributed much original work to the field. It deserves thorough study by all applied mathematicians, physicists, chemists and engineers working in the field or desiring a comprehensive introduction to it.

W. H. Stockmayer (Cambridge, Mass.).

***Kachanov, L. M. On the mechanics of plastic solids.** Plastic Deformation, Principles and Theories, edited and introduced by Henry H. Hausner, pp. 7–12. Mapleton House, Brooklyn, N. Y., 1948. $8.00.

Translated from J. Appl. Math. Mech. [Akad. Nauk SSSR. Zhurnal Prikl. Mat. Mech.] **4**, no. 3, 37–42 (1940). The author's initials are misprinted L. N. in this translation.

Kačanov, L. M. The plastic torsion of circular rods of variable diameter. Akad. Nauk SSSR. Prikl. Mat. Meh. **12**, 375–384 (1948). (Russian)

Using stress-strain relations of the deformation type for a work-hardening plastic material, the author discusses the torsion of a shaft of variable diameter. The differential equation of the stress-function is established (Monge-Ampère type) and the equivalent variational principle is discussed. Discussed in detail are the cases of (1) a nearly cylindrical shaft, (2) a thin-walled hollow shaft of variable diameter, (3) a thick-walled hollow shaft which does not deviate much from a cylindrical tube. The general equations are not evaluated to the extent where it could be readily decided whether A. A. Ilyushin's condition for the applicability of the deformation type of stress-strain law [cf. the third following review] is at least approximately satisfied. However, since this condition is known to be satisfied for the torsion of a solid cylinder or a cylindrical tube, the results obtained for the cases (1) and (3) may be expected to constitute reasonably good approximations. The stress distribution in case (2), on the other hand, is statically determintte and hence independent of the stress-strain law. *W. Prager* (Providence, R. I.).

Rabotnov, Yu. N. On a disk of equal resistance. Akad. Nauk SSSR. Prikl. Mat. Meh. **12**, 463–464 (1948). (Russian)

The paper is concerned with the problem of determining the profile of a rotating circular disk made of an elastic-plastic material in such a manner that the octahedral shearing stress has a constant assigned value throughout the disk. The treatment is based on a stress-strain law of the flow type. [As A. A. Ilyushin has shown [cf. the second following review], the use of such a stress-strain law is justified only if the directions of the principal axes of stress as well as the ratios of the principal components of the stress deviation are kept constant during the loading of a generic element of the material. While the first condition is fulfilled here on account of the rotational symmetry no proof is offered by the author that the second condition is fulfilled too.] *W. Prager* (Providence, R. I.).

Gleyzal, A. Plastic deformation of a circular diaphragm under pressure. J. Appl. Mech. **15**, 288–296 (1948).

***Gleyzal, A. N. Plastic deformation of a thin circular plate under pressure.** Plastic Deformation, Principles and Theories, edited and introduced by Henry H. Hausner, pp. 161–190. Mapleton House, Brooklyn, N. Y., 1948. $8.00.

Using stress-strain relations of the deformation type for an incompressible, plastic material with work-hardening, the author discusses the plastic equilibrium of a thin circular diaphragm under uniform pressure. The relevant differential equations are established and integrated numerically. The result is found to be in good agreement with experimental data. [No attempt is made to justify the use of stress-strain relations of the deformation type. As A. A. Ilyushin has shown [cf. the following review], such relations should be used only if the directions as well as the ratios of the principal strains remain constant at each point of the incompressible material. The first condition is satisfied here on account of the rotational symmetry. The second condition, however, is not fulfilled as can be seen from figs. 6 and 7: for $r=0.6a$ (a = radius of diaphragm), for instance, the ratio of radial to circumferential strain decreases considerably with increasing pressure.] *W. Prager* (Providence, R. I.).

***Ilyushin, A. A. Relation between the theory of Saint Venant-Levy-Mises and the theory of small elastic-plastic deformations.** Plastic Deformation, Principles and Theories, edited and introduced by Henry H. Hausner, pp. 97–116. Mapleton House, Brooklyn, N. Y., 1948. $8.00.

Translated from Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **9**, 207–218 (1945); these Rev. **7**, 144.

***Ilyushin, A. A. Some problems in the theory of plastic deformations.** Plastic Deformation, Principles and Theories, edited and introduced by Henry H. Hausner, pp. 45–96. Mapleton House, Brooklyn, N. Y., 1948. $8.00.

Translated from Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **7**, 245–272 (1943); these Rev. **6**, 252.

***Ilyushin, A. A. The theory for small elastic-plastic deformations.** Plastic Deformation, Principles and Theories, edited and introduced by Henry H. Hausner, pp. 117–136. Mapleton House, Brooklyn, N. Y., 1948. $8.00.

Translated from Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **10**, 347–356 (1946); these Rev. **8**, 240.

***Beliaev, N. M. Theories of plastic deformation.** Plastic Deformation, Principles and Theories, edited and introduced by Henry H. Hausner, pp. 13–44. Mapleton House, Brooklyn, N. Y., 1948. $8.00.

Translated from Izvestiya Akad. Nauk SSSR. Otd. Tehn. Nauk **1937**, 49–70.

Dorn, J. E., and Latter, A. J. Stress-strain relations for finite elastoplastic deformations. J. Appl. Mech. **15**, 234–236 (1948).

The authors propose the following specification of finite strain: three normal strains defined as the natural logarithms of the ratios of final to original lengths of line elements originally parallel to the coordinate axes, and three shearing strains defined as the cosines of the final angles between these line elements. Using this specification of finite strains, the authors formulate a stress-strain law of the flow type. *W. Prager* (Providence, R. I.).

Neuber, H. Allgemeine Lösung des ebenen Plastizitätsproblems für beliebiges isotropes oder anisotropes Fliessgesetz. Z. Angew. Math. Mech. **28**, 253–257 (1948). (German. Russian summary)

The author discusses the two-dimensional problems of plasticity (plane stress or plane strain) which are governed by the two equations of equilibrium and an arbitrary yield condition. The method of characteristics is used. If the significant stress components are denoted in the usual manner by σ_x, σ_y, τ_{xy} and if $p=\sigma_x+\sigma_y$, $\xi=\sigma_x-\sigma_y$, $\eta=2\tau_{xy}$,

then $dp^2 = d\xi^2 + d\eta^2$ along the characteristics of either family. Since the yield condition establishes a relation between p, ξ and η, the characteristics can be traced in the (ξ, η)-plane (stress deviation plane) as well as in the (x, y)-plane (physical plane). The mapping of the (ξ, η)-plane on the (x, y)-plane is studied and it is shown that, at corresponding points, the angle between the ξ-axis and a characteristic of one family in the (ξ, η)-plane equals twice the angle between the x-axis and the characteristic of the other family in the (x, y)-plane. *W. Prager* (Providence, R. I.).

Bohnenblust, H. F., and Duwez, Pol. Some properties of a mechanical model of plasticity. J. Appl. Mech. **15**, 222–225 (1948).

The mechanical model of a strain hardening material discussed in the paper consists of a large amount of elastic-perfectly plastic elements each of which is characterized by its yield stresses in tension and compression or, what amounts to the same, the elastic strains (ϵ and δ) corresponding to these yield stresses. The mechanical behavior of the model can then be expressed in terms of the distribution $p(\epsilon, \delta)$ of the elements over the values of ϵ and δ. It is assumed that $p(\epsilon, \delta) = p(\delta, \epsilon)$, i.e., that "in their totality the elements do not distinguish between elongation and contraction, although any particular element may show a preference for one direction." The stress-strain relations for loading and unloading are discussed; it is shown that the first does not completely determine the second, but only furnishes upper and lower bounds for it. These bounds are used to establish bounds for the potential energy. Experiments of G. I. Taylor and his associates are evaluated in the light of this theory and found to be consistent with it.
W. Prager (Providence, R. I.).

Philippidis, A. H. The general proof of the principle of maximum plastic resistance. J. Appl. Mech. **15**, 241–242 (1948).

Accepting Hencky's stress-strain relations for an incompressible, perfectly plastic material [Z. Angew. Math. Mech. **4**, 323–334 (1924)] and assuming [tacitly] that the yield limit is reached at every point of the body under consideration, the author proves a variational principle which may be considered as a generalization of Sadowsky's principle of maximum plastic resistance [same J. **10**, A-65–A-68 (1943); these Rev. **4**, 263]. If, for instance, the displacements are prescribed on the surface of the body, the author's principle states that the work of the surface tractions on the prescribed displacements is stationary when the actual state of stress is compared to any neighboring state of stress which also satisfies the equations of equilibrium and the yield condition. The reviewer [same J. **10**, A-238–A-239 (1943)], and G. H. Handelman [Quart. Appl. Math. **1**, 351–353 (1944); these Rev. **5**, 252] in special cases and more recently R. Hill [Quart. J. Mech. Appl. Math. **1**, 18–28 (1948); these Rev. **9**, 635] in the general case have discussed Sadowsky's principle using the stress-strain relations of v. Mises [Nachr. Ges. Wiss. Göttingen. Math.-Phys. Kl. 1913, 582–592]. If here the velocities are prescribed on the surface, the actual state of stress leads to a greater rate of work of the surface tractions on the prescribed velocities than any other state of stress which also satisfies the equations of equilibrium and the yield condition. Since Sadowsky uses the term "maximum effort" without giving a precise definition, either generalization of his principle appears legitimate.] *W. Prager* (Providence, R. I.).

Roderick, J. W. Theory of plasticity—elements of simple theory. Philos. Mag. (7) **39**, 529–539 (1948).

Simple bending and bending combined with axial loading are discussed for beams made of an elastic-plastic material which exhibits upper and lower yield limits.
W. Prager (Providence, R. I.).

Oldroyd, J. G. Rectilinear plastic flow of a Bingham solid. III. A more general discussion of steady flow. Proc. Cambridge Philos. Soc. **44**, 200–213 (1948).

[For parts I and II cf. same Proc. **43**, 396–405, 521–532 (1947); these Rev. **8**, 612; **9**, 121.] The paper is concerned with the steady rectilinear flow of a Bingham solid when the pressure remains constant along each stream line. The lines of constant velocity and their orthogonal trajectories are used as the coordinate lines of a curvilinear coordinate system; the equations of motion are transformed to this coordinate system and simple solutions are discussed.
W. Prager (Providence, R. I.).

Oldroyd, J. G. Rectilinear plastic flow of a Bingham solid. IV. Nonsteady motion. Proc. Cambridge Philos. Soc. **44**, 214–228 (1948).

[Cf. the preceding review.] The author establishes the basic equations for the unsteady rectilinear flow of a Bingham solid and interprets these equations in terms of an analogous heat conduction problem. The motion of the surfaces which separate plastic from elastic regions is investigated. The author treats as a first example the flow of a Bingham solid between two parallel plates one of which is at rest while the other is uniformly accelerated. A second example is concerned with the initial flow of a Bingham solid in a pipe of circular cross section in the case where the solid is suddenly subjected to a pressure gradient.
W. Prager (Providence, R. I.).

Krall, G., e Caligo, D. Moltiplicatore critico λ_σ di una distribuzione di carico su una vôlta autoportante. II. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **4**, 15–21 (1948).

For part I cf. Krall, same Rend. Cl. Sci. Fis. Mat. Nat. (8) **1**, 1281–1294 (1946); these Rev. **8**, 548.

Vasilesco, Florin. Recherches sur le flambement des poutres droites à section constante et à moment d'inertie variable. Ann. Sci. École Norm. Sup. (3) **64** (1947), 247–274 (1948).

Beams of uniform sections but variable moment of inertia are loaded by end thrust and (in some cases) lateral forces and end moments. The conventional questions concerning the buckling of such beams are asked. The attack is less efficient and has no more accuracy than the conventional Fredholm equation approach. *G. F. Carrier.*

Weigand, A. The problem of torsion in prismatic members of circular segmental cross section. Tech. Memos. Nat. Adv. Comm. Aeronaut., no. **1182**, 27 pp. (1948).

The original appeared in Luftfahrtforschung **20**, 333–340 (1944); these Rev. **5**, 252.

Parkus, H. Die Torsion geschlitzter Hohlwellen. Österreich. Ing.-Arch. **2**, 372–376 (1948).

Für das dünnwandige, tordierte Rohr mit n ($n = 2, 3, 4, \cdots$) gleichmässig über den Umfang verteilten Längsschlitzen wird die Spannungserhöhung durch die Schlitze gegenüber dem ungeschwächten Rohrquerschnitt berechnet. Das Verhältnis der beiden Spannungen beträgt mindestens 1.5 und

lässt sich für die praktisch in Betracht kommenden Abmessungen aus einer einfachen Formel bestimmen.

Author's summary.

Hruška, Václav. La chaînette élastique. Acad. Tchèque Sci. Bull. Int. Cl. Sci. Math. Nat. **45** (1944), 85–114 (1945).

Consider an elastic cord which is suspended from two points and which is acted upon by a gravitational field and an arbitrarily prescribed vertical load. The paper tabulates many theorems as to the symmetry, differentiability, monotonicity, etc., of the parameters associated with the elastic curve. The suspension bridge cable and the numerical evaluation of particular solutions are discussed briefly.

G. F. Carrier (Providence, R. I.).

Hruška, Václav. The elastic catenary. Rozpravy II. Třídy České Akad. **54**, no. 10, 83 pp. (1944). (Czech)

This paper contains the same results as that of the preceding review. Here, however, the proofs of the theorems are given. *G. F. Carrier* (Providence, R. I.).

Sengupta, H. M. On the bending of an elliptic plate under certain distributions of load. Bull. Calcutta Math. Soc. **40**, 17–35 (1948).

The biharmonic function is found which represents the deflection of an elliptic plate under a concentrated load placed along the major axis. This function is represented by certain singular terms plus a series of products of trigonometric and hyperbolic functions in the elliptic coordinate variables. *G. F. Carrier* (Providence, R. I.).

Mincberg, B. L. A mixed boundary problem of the theory of elasticity for a plane with a circular hole. Akad. Nauk SSSR. Prikl. Mat. Meh. **12**, 415–422 (1948). (Russian)

The general solution of a mixed two-dimensional boundary value problem of elasticity for an infinite plate with a circular hole is indicated. The method is identical with that proposed by N. Muschelišvili for the treatment of the first and second boundary value problems. The displacements are specified on a portion L_1 of the boundary of the circle and stresses are assigned on the remaining part L_2. The resultant force acting on L_1 is assumed to be known as well as the principal stresses at infinity. Two special examples are considered: (a) the plate is subjected to a uniform tensile stress at infinity, while L_2 is free and L_1 is allowed to undergo a rigid displacement; (b) L_2 is free and a rigid body subjected to a radial force presses along L_1; the forces at infinity are absent. *I. S. Sokolnikoff.*

Saharov, I. E. The bending of a wedge-shaped clamped plate under the action of an arbitrary load. Akad. Nauk SSSR. Prikl. Mat. Meh. **12**, 407–414 (1948). (Russian)

The paper deals with small deflections w of an infinite wedge-shaped elastic plate subjected to an arbitrary normal load $p(\rho, \theta)$, where ρ and θ are polar coordinates in the plane of the plate. If the angle of the wedge is 2α, the clamping condition requires that on the sides $\theta = \pm\alpha$ of the wedge $w = \rho^{-1}\partial w/\partial\theta = 0$. The solution is sought in the form $w = (p/8\pi D)(r^2\Gamma + \psi)$, where ψ is a regular biharmonic function, $\Gamma(\rho, \theta; \rho_0, \theta_0)$ is a harmonic Green's function for the wedge, $r^2 = \rho^2 + \rho_0^2 - 2\rho\rho_0 \cos(\theta - \theta_0)$, p is a concentrated load applied at (ρ_0, θ_0) and D is flexural rigidity. The solution is obtained, for the deflection w under a concentrated load p, in the form of an integral with ρ_0 and θ_0 as parameters. The problem for a distributed load is solved by integrating over the loaded region. The paper contains calculations for the deflection, and bending and twisting moments along the sides of the plate, under a concentrated load when $\alpha = \pi/4$. *I. S. Sokolnikoff* (Los Angeles, Calif.).

Seth, B. R. Bending of rectilinear plates. Bull. Calcutta Math. Soc. **40**, 36–40 (1948).

The author relates the deflection of a simply supported plate to the torsion problem for the same boundary and proceeds to obtain that deflection in what is essentially the conventional product-series solution. *G. F. Carrier.*

Föppl, L. Die schiefe Kreiskegelschale bei zwei Beanspruchungsarten. Z. Angew. Math. Mech. **24**, 195–203 (1944).

In a previous paper [S.-B. Math.-Nat. Abt. Bayer. Akad. Wiss. **1944**, 1–17 (1944); these Rev. **9**, 547] the author obtained the membrane stress distribution in an inclined conical shell with equation $axz = x^2 + y^2$, subjected to a uniform normal pressure. In the present paper the same problem is solved for the more general case where the equation of the shell is assumed in the form $(ax - bz)z = x^2 + y^2$. Numerical calculations on the basis of explicit results show the dependence of the stress distribution on the value of the parameter b/a. The second kind of load application considered is that of a concentrated load at the apex of the shell. *E. Reissner* (Cambridge, Mass.).

Sauer, Robert. Geometrische Bemerkungen zur Membrantheorie der negativ gekrümmten Schalen. Z. Angew. Math. Mech. **28**, 198–204 (1948). (German. Russian summary)

It is shown that the characteristics of the differential equations of equilibrium of the linear membrane theory (momentless theory) of thin shells are the asymptotic lines of the shell surface. This confirms a result of W. Flügge [same Z. **25/27**, 65–70 (1947); these Rev. **9**, 316] regarding the reality of the characteristics for shells with negative curvature, in contrast to what is the case for shells with positive curvature. On the basis of this result there is outlined a procedure for the graphical determination of the state of stress in shells with negative curvature, in the absence of surface loads. Use is made of results by W. Blaschke [Proc. Fifth Intern. Congress Math., Cambridge, 1912, vol. 2, pp. 291–294 (1913)] concerning graphical statics of membranes. Explicit solutions are obtained for the stresses in shell surfaces of the second degree, without surface loads. *E. Reissner* (Cambridge, Mass.).

Reissner, Eric. Corrections to the paper "Stresses and small displacements of shallow spherical shells. II." J. Math. Physics **27**, 240 (1948).

The paper appeared in the same J. **25**, 279–300 (1947); these Rev. **8**, 359.